MATHEMATICS

STUDENT'S TEXTBOOK GRADE 12

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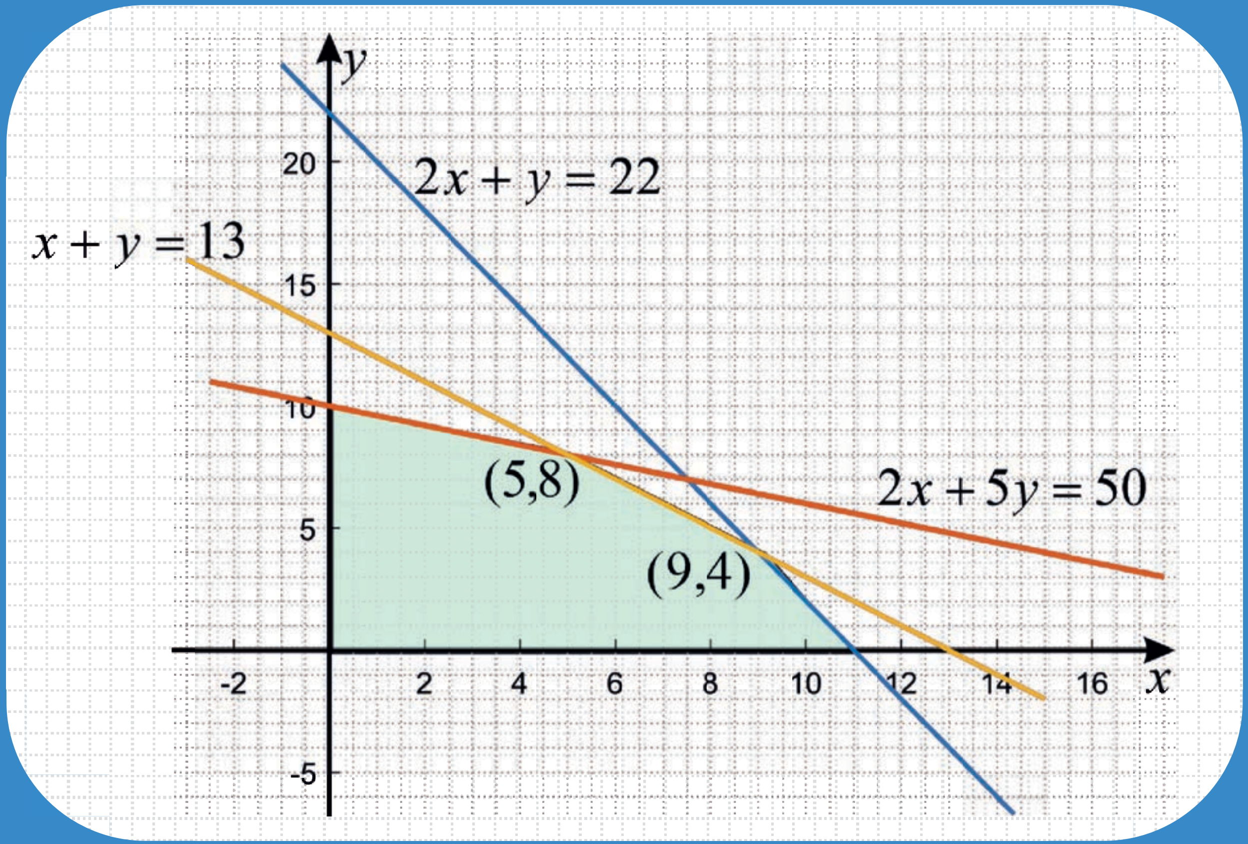

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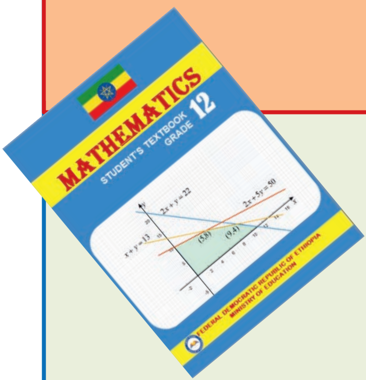

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MATHEMATICS

STUDENT'S TEXTBOOK GRADE 12

Writers:

Habtamu Garoma (PhD)

Wesen Legesse (PhD)

Editors:

Zewdu Desalegn (MSc) (Content Editor)

Solomon Melesse (Prof., PhD) (Curriculum Editor)

Melaku Wakuma (PhD) (Language Editor)

Illustrator:

Zerihun Kinfe (PhD)

Designer:

Berie Getie (MSc)

Evaluators:

Dawit Ayalneh Tebekew (MSc)

Matebie Alemayehu Wasihun (MA)

Mustefa Kedir Edao (BEd)

Tesfaye Sileshi (MA)

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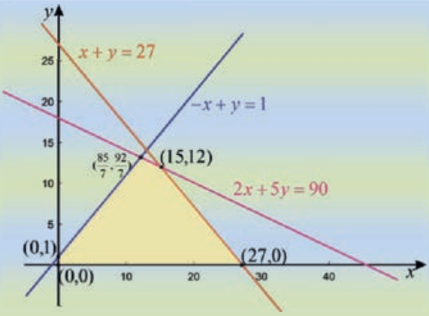
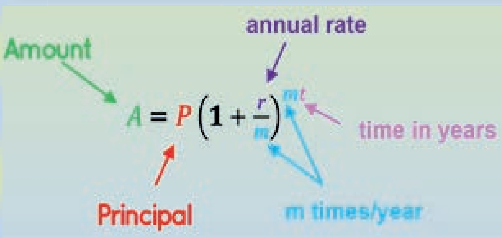
Table of Contents

Table of Contents

Unit 4 INTRODUCTION TO LINEAR PROGRAMMING 237



Unit 5 MATHEMATICAL APPLICATIONS IN BUSINESS 303



Welcoming Message to Grade 12 Students

Dear students, you are welcome to Grade 12 Mathematics educations. This is a golden opportunity in your academic career. Joining this grade level is a new experience and a transition to tertiary level mathematics educations. At this grade level, you are expected to receive new and advanced opportunities that can help you learn and grow in the field of Mathematics, in life and work. Enjoy it!

Introduction about the Students' Textbook

Dear student, this textbook is organized in such a way that there is an introductory remark about how to use the textbook, some suggestions about how to care the textbook so that it will serve for many beneficiaries for a long period of time, and basic units are addressed after the introduction section.

The Grade 12 textbook is organized having 5 units. The units include: Sequence and Series, Introduction to Pre-calculus, Statistics, Introduction to Linear Programming, and Mathematical Applications in Business**.** Each unit is composed of an introduction, learning outcome, lessons, key words, and summary and review exercise.

In a unit, a brief introduction about the unit is provided. Following a brief introduction of a unit, the expected learning outcomes are presented. Once the objectives are communicated, the different lessons of the unit are presented. In the end, summary and review exercise are packed, respectively.

As mentioned earlier, a unit in the textbook is divided into different lessons. In every lesson of each unit, you learn about mathematics frequently through five components. That is, structurally, the lessons in this textbook usually have five components; namely, Activity, Definitions/Theorems/Notes, Examples and Solutions, and Exercises.

The most important part in this process is to practice problems by yourself based on what your teacher shows and explains. Your teacher will also give you feedback and assistance, and facilitate further learning. In such a way, you will be able not only acquire new knowledge and skills but also develop them further. Herewith, a brief explanation of each sub-component of a lesson is forwarded.

Activity

This part of the lesson requires you to revise what you have learnt at different stages of your mathematics education and reflect on the topic under discussion by using this background knowledge. The activity also introduces you to what you are going to learn in the respective new lesson or topic.

Definition/Theorem/Note

This part of the textbook presents and explains new concepts/ definitions/theorems.

Example and Solution

Here, the textbook provides you specific examples and thereby helps you to improve your understanding of the new content. In this part, your teacher will give you explanations and/or solutions and you are, therefore, advised to listen to your teacher's explanations very carefully and participate actively in the process. You are also required to refer to the solution part of the textbook for your review and self-learning. In this sub-section, a student is further expected to study on the remaining examples that a classroom teacher will not explain during his/her classroom instruction.

Exercise

Under this part of the textbook, you will be required to solve the problems or questions that are given as exercise individually, in pairs or groups so as to further practice what you have learnt in the examples. That is, a teacher may provide selected exercise questions for you as class work, homework or project work. When you are doing the exercises either individually, in pairs or groups, you are expected to share your opinions with your friends, listen to others' ideas carefully and compare your ideas with others. However, it is always advisable to make some efforts individually before working in pair or groups in order to have a better understanding and input in the process.

In some of the cases, some exercises may not be covered by a classroom teacher. In that case, a student is expected to study the remaining items of the exercise a teacher will not cover during classroom instruction. A teacher will provide the possible answers for those items in the exercises after completing your study, and you are expected to check what you will do against the respective teacher's answers.

Checking in every step

Throughout a lesson, students shall check their progress and the respective teacher should do the same. It is also true that students are required to check the correct answers and solutions for the exercises at the end of each unit. If you want to practice more, therefore, you need to go to the review exercises. Doing the Review Exercise is always recommended since they help you develop a concrete view of the lesson contents.

Generally, students are advised to practice, drill and exercise each activity, example, and exercise on a daily basis as mastering mathematical skills take a long period of time and frequent practice to be second nature. You are also requested to read the notes with deep understanding.

UNIT 1

SEQUENCES AND SERIES

Unit Outcomes

By the end of this unit, you will be able to:

- Understand sequence and series.
- Compute terms of a sequence from a given rule.
- Use given terms to develop a formula that represent the sequence.
- Identify different types of sequences and series.
- Compute the partial and infinite sum of some sequences.
- Apply your understanding the knowledge of sequences and series to real-life problems.

Unit Contents

KEY Words

- common ratio
- convergent series
- divergent series
- arithmetic sequence
- partial sums
- general term
- geometric sequence
- infinite sequence
- infinite series
- finite sequence
- common difference
- sequence
- series
- terms of a sequence
- recursion formula

Introduction

Mathematics has an enormous number of uses in our daily life. There is, in fact, no area of life that is not affected by mathematics. For instance, civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programs useful. Mathematics is present everywhere from distance, time and money to art, design and music.

A sequence is an arrangement of numbers in a definite order according to some rule. Sequences and Series play an important role in various aspects of our lives. They help us to predict, evaluate and monitor the outcome of a situation or event and help us in decision making.

1.1 Sequences

Activity 1.1

1. People wants to plant trees in a certain pattern in the green area of a community like 20 plants in the first row, 34 plants in the second row and 48 plants in the third row, and so on. How many trees people will plant in the 5^{th} and 6^{th} row?
2. Find the next two terms of the following sequence.

   a. $\{50, 47, 44, 41, ...\}$ b. $\{2, 12, 22, 32, ...\}$

Definition 1.1

A sequence is a function whose domain is the collection of all integers greater than or equal to a given integer m (usually 0 or 1). A sequence is usually denoted by a_n. The functional values: $a_1, a_2, a_3, \ldots, a_n, \ldots$ are called the terms of a sequence, and a_n is called the general term, or the n^{th} term of the sequence. There are two types of sequences depending on its last term.

Finite Sequence: A sequence that has a last term. The domain of a finite sequence is $1, 2, 3, \ldots, n$.

Infinite sequence: A sequence that does not have a last term. The domain of an infinite sequence is the set of natural numbers $(\mathbb{N})$.

Example 1

List the first five terms of each of the sequences whose general terms are given below where n is a positive integer.

$(a)\ a_n = 2n-1 \qquad (b)\ a_n = \left(-\frac{1}{3}\right)^{n-1} \qquad (c)\ a_n = \frac{1}{n}$

Solution

$(a)\ a_n = 2n-1:$

$$a_1 = 2(1)-1 = 1, \quad a_2 = 2(2)-1 = 3, \quad a_3 = 2(3)-1 = 5,$$
$$a_4 = 2(4)-1 = 7, \quad a_5 = 2(5)-1 = 9.$$

Therefore, the first five terms are $1, 3, 5, 7$ and 9.

$(b)\ a_n = \left(-\frac{1}{3}\right)^{n-1}$

$$a_1 = \left(-\frac{1}{3}\right)^{1-1} = 1, \quad a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}, \quad a_3 = \left(-\frac{1}{3}\right)^{3-1} = \frac{1}{9},$$
$$a_4 = \left(-\frac{1}{3}\right)^{4-1} = -\frac{1}{27}, \quad a_5 = \left(-\frac{1}{3}\right)^{5-1} = \frac{1}{81}.$$

Therefore, the first five terms are 1, $-\frac{1}{3}$, $\frac{1}{9}$, $-\frac{1}{27}$ and $\frac{1}{81}$.

(c) $a_n = \frac{1}{n}$:

$$a_1 = \frac{1}{1} = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{4},\ a_5 = \frac{1}{5}.$$

Therefore, the first five terms are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$.

Example 2

Draw the graph of sequence $a_n = 2n+1$.

Solution

Make a table with n and a_n, as follows. Then plot each ordered pair (n, a_n).

n	1	2	3	4	5
a_n	3	5	7	9	11

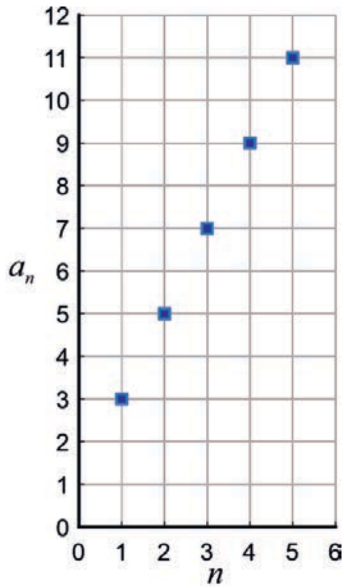


Figure 1.1

Note

From the Figure 1.1, we can observe that the graph of the sequence follows the pattern of a linear equation.

Exercise 1.1

1. List the first five terms of each of the sequences whose general terms are given below where n is a positive integer:

   a. $a_n = 2n$ b. $a_n = \left(\frac{1}{2}\right)^{n-1}$ c. $a_n = \frac{n}{n+1}$ d. $a_n = n^3$

2. Draw the graph of the following sequences and observe the pattern of the sequence.

   a. $a_n = 3n-1$, b. $a_n = \frac{1}{n}$, c. $a_n = (-1)^n$.

3. Bontu's uncle gave 130 Ethiopian birr to her in January, in the next month she saves money and has 210 Ethiopian birr and in the third month she has 290 Ethiopian birr. How much money will she have in the fourth, fifth, sixth and seventh month respectively.

Fibonacci and Mulatu sequences

Activity 1.2

Two squares with a side length of 1 are arranged side by side. The rectangle made has a vertical length of 1 and a horizontal length of 2. Place a square with a side length of 2 next to it. Then, the rectangle made has vertical length is 3, horizontal length is 2. After this, arrange a square with a side length of 3, a square with a length of 5, and a square with a length of 8 to make a large rectangle.

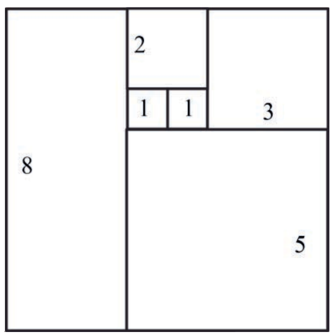


Figure 1.2

Based on Figure 1.2 above, answer the following questions:

a. What is the sum of the area of all the inner squares?

b. What is the area of the outer rectangle?

c. What is the relation between your answer to parts a and b?

Fibonacci's sequence

HISTORICAL NOTE

Leonardo Fibonacci (circa 1170, 1240)

Italian mathematician Leonardo Fibonacci made advances in number theory and algebra. Fibonacci, also called Leonardo of Pisa, produced numbers that have many interesting properties such as the birth rates of rabbits and the spiral growth of leaves on some trees.

He is especially known for his work on series of numbers, including the Fibonacci series. Each number in a Fibonacci series is equal to the sum of the two numbers that came before it. Fibonacci sequence arose when he was trying to solve a problem of the following kind concerning the breeding of rabbits.

"Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at the age of two months. If we start with one new born pair, how many pairs of rabbits will we have in the n^{th} month?"

Fibonacci sequence is defined as: $F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$

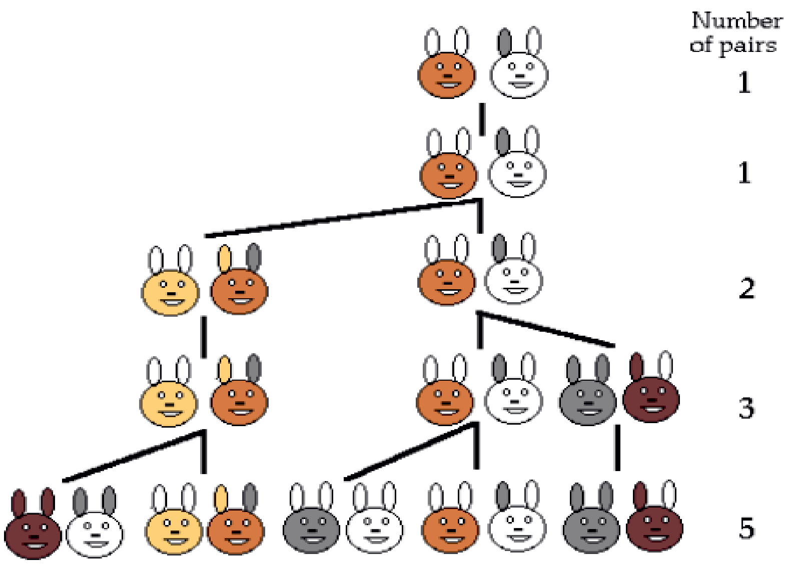


Figure 1.3

Example

List the first eight terms of the Fibonacci sequence and draw its graph.

Solution

Make a table with n and a_n, as follows. Then plot each ordered pair (n, F_n).

n	0	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	34	55

So, the graph becomes

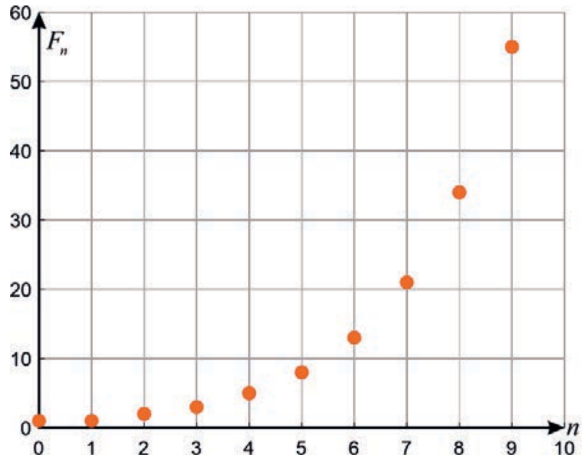


Figure 1.4

Mulatu sequence

HISTORICAL NOTE

Mulatu Lemma

Professor Mulatu Lemma is an Ethiopian Mathematician.

He pursued an education through a university in Ethiopia, earning a Bachelor of Science in 1977 from Addis Ababa University.

He continued his studies at the aforementioned university and obtained a Master of Science in applied mathematics in 1982. Following these accomplishments, he was enrolled at Kent State University, where he pursued a Master of Arts in pure mathematics in 1993. Prof. Mulatu completed his academic journey with a Doctor of Philosophy from Kent State University in 1994. In 2011, he introduced the Mulatu's Number (named after him) to the mathematical community and to the world.

Professor Mulatu introduced a sequence of the form:

$$M_n = \begin{cases} 4 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ M_{n-1} + M_{n-2} & \text{if } n \geq 2. \end{cases}$$

Recursive Sequence

A sequence that relates to the general term a_n of a sequence where one or more of the terms that comes before it is said to be defined recursively. The domain of recursive sequence can be the set of whole numbers. For Example, Mulatu and Fibonacci sequences are some of the examples of recursion formula.

Exercise 1.2

1. Find the 12^{th} term of the Fibonacci's sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$.
2. List the first eight terms of Mulatu's sequences and draw its graph.

1.2 Arithmetic and Geometric Sequences

1.2.1 Arithmetic Sequences

Activity 1.3

Find the difference between consecutive terms for each of the following sequences.

a. 3, 7,11,15, ...   b. 3, −1,−5,−9,...   c. $2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$

Definition 1.2

Arithmetic sequence or arithmetic progression is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term. The fixed number is called common difference of the sequence.

Example 1

For the following arithmetic sequence 2, 5, 8, 11, 14,…, what is the first term, third term and common difference? Find the 6th term.

Solution

$$2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14$$

1st term: 2, 3rd term: 8

The common difference is 3.

6th term: 5th term + 3 = 14 + 3 = 17.

Example 2

Given that the 1st term of an arithmetic sequence is 10, and the common difference is -4, find the terms from 2nd to 5th term.

Solution

2^{nd} term: 1^{st} term + (-4) = 10 + (-4) = 6.

3^{rd} term: 2^{nd} term + (-4) = 6 + (-4) = 2.

4^{th} term: 3^{rd} term + (-4) = 2 + (-4) = -2.

5^{th} term: 4^{th} term + (-4) = -2+ (-4) = -6.

Therefore, the terms from 2^{nd} to 5^{th} term are 6, 2, -2, -6.

Exercise 1.3

1. For the following arithmetic sequences, what are the first term, third term and common difference? Find the 6^{th} term.

   a. {3, 5, 7, 9, 11...}

   b. {9, 6, 3, 0, -3…}

2. Find the terms from 2^{nd} to 5^{th} of an arithmetic sequence when the

   a. 1st term is 1, and the common difference is 5.

   b. 1st term is 2, and the common difference is $-\frac{1}{2}$.

General term of arithmetic sequence

From the Activity 1.3, (a), 3, 7, 11, 15, 19,… the first term is 3, and the common difference is 4. Let the common difference be d, and let A_n be the n^{th} term of the sequence, then,

$$A_1 = 3$$
$$A_2 = 3+4 = A_1 + d$$
$$A_3 = 3+4+4 = A_1 + 2d$$
$$A_4 = 3+4+4+4 = A_1 + 3d$$
$$\cdots$$
$$A_n = 3+4+4+4+\ldots+4 = A_1 + (n-1)d$$

Suppose $A_n : A_1, A_2, A_3, A_4, A_5, A_6, \ldots$ is an arithmetic sequence.

$$A_2 = A_1 + d$$
$$A_3 = A_2 + d = A_1 + d + d = A_1 + 2d$$
$$A_4 = A_3 + d = A_1 + 2d + d = A_1 + 3d$$

$$A_5 = A_4 + d = A_1 + 3d + d = A_1 + 4d$$
$$A_6 = A_5 + d = A_1 + 4d + d = A_1 + 5d$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$A_n = A_{n-1} + d = A_1 + (n-2)d + d = A_1 + (n-1)d$$

Thus, the following formula is derived:

Theorem 1.1

If A_n is an arithmetic sequence with the first term A_1 and a common difference d, then the n^{th} term of the sequence is given by $A_n = A_1 + (n-1)d$.

Example 1

Find the general term of the sequence A_n, when the first term is 5, and the common difference is 4.

Solution

The given sequence is arithmetic sequence with $A_1 = 5$ and $d = 4$.

From the theorem 1.1, we have the formula $A_n = A_1 + (n-1)d$. So,

$A_n = A_1 + (n-1)d = 5 + (n-1)(4) = 5 + 4n - 4 = 4n + 1$.

Example 2

What is the 31^{st} term of the sequence $1, 4, 7, 10, ...$?

Solution

The given sequence is arithmetic sequence with $A_1 = 1$ and $d = 3$

$A_n = A_1 + (n-1)d = 1 + 3(n-1) = 3n - 2$.

Then,

$A_{31} = 3(31) - 2 = 93 - 2 = 91$.

Exercise 1.4

1. Find the general term of the sequence A_n, when

   a. $A_1 = 2, \ d = 3.$ b. $A_1 = 10, \ d = -5.$

2. What is the 10^{th} term of the sequence, 10, 6, 2, -2, …?

Further on arithmetic sequences

Example 1

When the third term is 10 and the sixth term is 1,

a. Find the general term of sequence A_n.

b. Find A_8.

Solution

a. Applying arithmetic sequence formula and substituting existing values yields:

$$A_3 = A_1 + (3-1)d, \qquad 10 = A_1 + 2d$$
$$A_6 = A_1 + (6-1)d, \qquad 1 = A_1 + 5d$$
$$\begin{cases} A_1 + 2d = 10 \\ A_1 + 5d = 1 \end{cases}$$

Subtracting these two equations,

$$3d = -9 \Rightarrow d = -3, \ A_1 + 2d = 10, \ A_1 + 2(-3) = 10, \ A_1 = 10 + 6 = 16.$$

Therefore, the general term becomes

$$A_n = A_1 + (n-1)d = 16 + (n-1)(-3) = 16 - 3n + 3 = 19 - 3n.$$

b. $A_n = 19 - 3n \Rightarrow A_8 = 19 - 3(8) = 19 - 24 = -5.$

Example 2

Determine whether or not the sequences with the following general terms are arithmetic. *a.* $a_n = 3n - 2$ *b.* $a_n = 3n^2 - 2$

Solution

a. To solve such types of problem, we have to show the difference between successive terms is constant.

$$a_n = 3n - 2$$
$$a_{n+1} = 3(n+1) - 2 = 3n + 1$$
$$a_{n+1} - a_n = 3n + 1 - (3n - 2) = 3.$$

Since, the difference between successive term is constant, it arithmetic sequence

b. $a_n = 3n^2 - 2$

$$a_n = 3n^2 - 2$$
$$a_{n+1} = 3(n+1)^2 - 2 = 3(n^2 + 2n + 1) - 2 = 3n^2 + 6n + 1$$
$$a_{n+1} - a_n = 3n^2 + 6n + 1 - (3n^2 - 2) = 6n + 3.$$

Since, the difference between successive terms is not constant, it is not arithmetic sequence.

Exercise 1.5

1. Find the general term of the arithmetic sequence A_n, when

   a. $A_4 = 15,\ \ A_8 = 27.$ b. $A_5 = 20,\ \ A_{10} = 0.$

2. Given arithmetic sequence with $A_2 = 3$ and $A_5 = 24$. Find A_n and A_{11}.

3. Determine whether or not the sequences with the following general terms are arithmetic.

   a. $a_n = 7n - 3$ b. $a_n = 5n - 3$ c. $a_n = n^2 + n + 1$ d. $a_n = 3^n$

Arithmetic mean between two numbers

The term(s) of arithmetic sequence that lie between two given terms are called the arithmetic mean.

Example 1

Given that $1, x, 8$ is an arithmetic sequence, find x.

Solution

Since it is arithmetic sequence, the difference between two consecutive terms is constant.

$$x - 1 = 8 - x$$
$$2x = 9$$
$$x = \frac{9}{2}$$

Example 2

The first and sixth terms of an arithmetic sequence are 4 and 29. Find the values of terms 2, 3, 4 and 5.

Solution

Let the four terms be A_2, A_3, A_4, A_5.

So, $4, A_2, A_3, A_4, A_5, 29$ form an arithmetic sequence. Then, $A_n = A_1 + (n-1)d$.

$$A_1 = 4$$
$$A_6 = A_1 + (6-1)d$$
$$29 = 4 + 5d$$
$$d = 5$$

Thus, $A_n = 4 + (n-1)5 = 5n - 1$

$A_2 = 4 + 5 \times 1 = 9$

$A_3 = 4 + 5 \times 2 = 14$

$A_4 = 4 + 5 \times 3 = 19$

$A_5 = 4 + 5 \times 4 = 24$.

Exercise 1.6

1. Given that the sequence $3, x, 7$ is an arithmetic sequence, find x.
2. Given that the sequence $\frac{1}{12}, \frac{1}{x}, \frac{1}{6}$ is an arithmetic sequence, find x.
3. Find the arithmetic mean of 4 and 14.
4. Insert four arithmetic means between 4 and 14 to create an arithmetic sequence.

1.2.2 Geometric Sequences

Activity 1.4

Find the ratio between the consecutive terms of each of the following sequences.

a. 2 ,6, 18, 54,...   *b.* $2, -2, 2, -2, \ldots$ *c.* $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

Definition 1.3

A geometric sequence or geometric progression is one in which the ratio between consecutive terms is a non-zero constant. This constant is called the common ratio.

$\{G_n\}$ is geometric sequence if and only if $r = \frac{G_{n+1}}{G_n}$, $r \in \mathbb{R}$ & $r \neq 0$ where r is the common ratio.

Example 1

For the following geometric sequence 3, 6, 12, 24, 48,…, find the common ratio, r, and the 6^{th} term.

Solution

The common ratio:

$$\frac{G_2}{G_1}=\frac{6}{3}=2,\ \frac{G_3}{G_2}=\frac{12}{6}=2,\ \frac{G_4}{G_3}=\frac{24}{12}=2,\ldots$$

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \cdots$$

Thus, $r=2$.

The 6^{th} term: $G_6=G_5r=48\times 2=96.$

Example 2

The 1^{st} term of a geometric sequence is $-\frac{1}{2}$, and its common ratio is $\frac{1}{2}$, find the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} term.

Solution

We are given $G_1=-\frac{1}{2}$ and $r=\frac{1}{2}$, then

$$G_2=G_1r=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}$$

$$G_3=G_2r=-\frac{1}{4}\times\frac{1}{2}=-\frac{1}{8}$$

$$G_4=G_3r=-\frac{1}{8}\times\frac{1}{2}=-\frac{1}{16}$$

$$G_5=G_4r=-\frac{1}{16}\times\frac{1}{2}=-\frac{1}{32}.$$

Exercise 1.7

1. For the geometric sequence 1, 2, 4, 8, 16…, find the common ratio, r, and the 6^{th} term.
2. Given that the 1^{st} term of a geometric sequence is 1, and its common ratio is 3, find the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} term.

Determining the n[th] term of geometric sequence

From the activity 1.4, (a), we have the geometric sequence 2, 6, 18, 54,… which has a common ratio of 3.

$$G_1 = 2$$
$$G_2 = 6 = 2\times 3 = G_1 \times 3^1$$
$$G_3 = 18 = 6\times 3 = G_2 \times 3 = G_1 \times 3\times 3 = G_1\times 3^2$$
$$G_4 = 54 = 18\times 3 = G_3\times 3 = G_1\times 3\times 3\times 3 = G_1\times 3^3$$
$$\cdots$$
$$G_n = G_1\times 3\times 3\times 3\times \ldots\times 3 = G_1\times 3^{n-1}$$

From Definition 1.3,

$$G_2 = G_1\times r$$
$$G_3 = G_2\times r = G_1\times r\times r = G_1\times r^2$$
$$G_4 = G_3\times r = G_1\times r^2\times r = G_1\times r^3$$
$$G_5 = G_4\times r = G_1\times r^3\times r = G_1\times r^4$$
$$\cdots \qquad \cdots$$
$$G_n = G_1\times r^{n-1}$$

Thus, the following theorem is deduced:

Theorem 1.2

If $\{G_n\}$ is a geometric sequence with the first term G_1 and common ratio r, then the n^{th} term of the sequence is given by $G_n = G_1 r^{n-1}$.

Example 1

Find G_n, when the first term is 3 and the common ratio is 2.

Solution

Given $G_1 = 3$ and $r = 2$, then

$$G_n = G_1 r^{n-1} = 3\times 2^{n-1}.$$

Example 2

Find the n^{th} term, G_n of the sequence $1, -2, 4, -8, 16, \ldots$.

Solution

$G_1 = 1$, $r = -2$

Applying the formula for the n^{th} term of a geometric sequence, $G_n = G_1 r^{n-1}$.

$$G_n = (-2)^{n-1}.$$

Example 3

Find the 6^{th} term of the geometric sequence whose first term is 1 and common ratio is 2.

Solution

Given $G_1 = 1$ and $r = 2$, then $G_n = G_1 r^{n-1} = 1 \times 2^{n-1} = 2^{n-1}$.

Therefore, $G_6 = 2^{6-1} = 2^5$ or 32.

Exercise 1.8

1. For each of the following, find the n^{th} term of the geometric sequence.

   a. $G_1 = 2,\ r = 5$
   b. $G_1 = 1,\ r = -3$
   c. $G_1 = 2,\ r = -2$
   d. $G_1 = -3,\ r = \frac{1}{2}$

2. Find the n^{th} term G_n of the following sequences

   a. $3, 6, 12, 24, \ldots$
   b. $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$
   c. $27, 9, 3, 1, \ldots$

3. Find the 5^{th} term of the geometric sequence whose first term is -3 and common ratio is $-\frac{1}{2}$.

Geometric mean between two numbers

When a, m, and b are terms in a geometric sequence, then m is called the geometric mean between a and b. $(a \neq 0,\ b \neq 0,\ m \neq 0)$. In a geometric sequence, the ratio between consecutive terms is constant.

$$r = \frac{m}{a} = \frac{b}{m}$$
$$m^2 = ab$$
$$m = \pm\sqrt{ab}$$

Example 1

When $2, x, 5, \ldots$ is a geometric sequence, find $x (x \neq 0)$.

Solution

As the ratio between the consecutive terms is the same,

$$\frac{x}{2} = \frac{5}{x},\ \ x^2 = 10,\ \ x = \pm\sqrt{10}.$$

Example 2

Find geometric mean between 2 and 8.

Solution

Let the geometric mean be $m (m \neq 0)$, then

$$r = \frac{m}{2} = \frac{8}{m}$$
$$m^2 = 2 \times 8$$
$$m = \pm\sqrt{16} = \pm 4.$$

Therefore, the geometric mean between 2 and 8 is -4 or 4.

Example 3

Find the 8^{th} term of the geometric sequence whose 1^{st} term is 5 and 4^{th} term is $\frac{1}{25}$.

(Express the answer in the form of exponent).

Solution

$G_1 = 5$ and $G_4 = \frac{1}{25}$,

$G_4 = G_1 r^3 = 5r^3 = \frac{1}{25}$.

Then, $r^3 = \left(\frac{1}{5}\right)^3$, therefore $r = \frac{1}{5}$.

$$G_8 = G_1 r^7 = 5\left(\frac{1}{5}\right)^7 = \frac{1}{5^6}.$$

Exercise 1.9

1. Find the geometric mean between 3 and 12.
2. In a geometric sequence, the 2^{nd} term is 12 and the 6^{th} term is 192. Find the 11^{th} term.
3. If $x, 4x+3, 7x+6$ are consecutive terms of a geometric sequence, find the value(s) of x, $x \neq 0$.
4. Find three consecutive terms of a geometric sequence, such that their sum is 35 and their product is 1000. Let the terms be $\frac{a}{r}$, a and ar. $(a \neq 0, r \neq 0)$

1.3 The Sigma Notation and Partial Sums

In the previous sections, you learned about the individual terms of a sequence. In this section, you will learn how to add the terms of a sequence, i.e. find the sum of the terms.

Partial sums

Given the sequence a_n

$S_1 = a_1$, S_1 is the first term of the sequence.

$S_2 = a_1 + a_2$, S_2 is the sum of the first two terms of the sequence.

$S_3 = a_1 + a_2 + a_3$, S_3 is the sum of the first three terms of the sequence.

$S_4 = a_1 + a_2 + a_3 + a_4$, S_4 is the sum of the first four terms of the sequence,

and so on.

$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, S_n is the sum of the first n terms of the sequence called the partial sum.

Definition 1.4

Let $\{a_n\}_{n=1}^{n=\infty}$ be a sequence. The sum of the first n terms of the sequence, denoted by S_n is called the partial sum of the sequence. Such summation is denoted as follows.

$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, where k is the index of the summation, 1 is the lower limit of summation, n is the upper limit of the summation and $\sum$ is the sigma notation or the summation notation.

Example 1

Find the sum of the first five even natural numbers.

Solution

$a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10$. Then,

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 2 + 4 + 6 + 8 + 10 \\ &= 30 \end{aligned}$$

Example 2

Let $a_n = 3n + 1$, find S_6

Solution

$$a_n = 3n + 1$$

$$a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, a_6 = 19$$

$$\begin{aligned} S_6 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 4 + 7 + 10 + 13 + 16 + 19 \\ &= 69 \end{aligned}$$

Example 3

Given the general term $a_n = \dfrac{1}{n(n+1)}$, find the sum of the first

a. 99 terms b. n terms

Solution

By using partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Solving for A and B gives $A = 1$ and $B = -1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$(a)\quad S_{99}=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+...+(\frac{1}{99}-\frac{1}{100})=1-\frac{1}{100}=0.99$

$(b)\quad S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-...+\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}=\frac{n}{n+1}$

Note: Such a sequence is said to be telescoping sequence.

Exercise 1.10

1. Find the sum of:
   a. the first five odd natural numbers.
   b. the first ten odd natural numbers.
2. Find the sums of the following sequences to the term given.
   a. $a_n=4n-3,\ S_5.$ b. $a_n=3-5n,\ S_8.$ c. $a_n=n^2+1,\ S_6.$
3. Given the general term $a_n=\frac{2}{n^2+5n+6}$, find the sum of the first n^{th} terms.

Sigma notation

Sigma notation is a method used to write out a long sum in a concise way. We use sigma notation for writing finite and infinite numbers of terms in a sequence. The sum is denoted by the sigma notation using the Greek letter $\sum$ (sigma).

Example 1

Express the following sigma notation in the form of the sum

a. $\sum_{k=1}^{8}3k$ b. $\sum_{k=2}^{6}k^2$

Solution

a. $\sum_{k=1}^{8}3k=3+6+9+12+15+18+21+24=108.$ b. $\sum_{k=2}^{6}k^2=4+9+16+25+36=90.$

Example 2

Which one of the following express the sum, $3^2+4^2+5^2+6^2+7^2$?

a. $\sum_{k=3}^{7} k^2$ b. $\sum_{i=3}^{7} i^2$ c. $\sum_{k=2}^{6} (k+1)^2$

Solution

All of them express the given sum.

Exercise 1.11

1. Express the following sigma notations in the form of a sum.

   a. $\sum_{k=1}^{6} 2k$ b. $\sum_{k=3}^{5} k^2$ c. $\sum_{k=1}^{n} 3^k$ d. $\sum_{k=3}^{5} k^3$

2. Express the following using the sigma notation

   $2^2+4^2+6^2+8^2+10^2+12^2$.

1.3.1 Sigma Notation

Properties of sigma notation

The sequence $\{a_n\}$, where all the terms are c, the sum of the first nth term is

$$\sum_{k=1}^{n} a_n = c+c+c+\ldots+c = nc$$

That is, $\sum_{k=1}^{n} a_n = nc$

In particular, $\sum_{k=1}^{n} 1 = n(1) = n$.

Properties of Sigma Notation

(1) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, c is a constant

(2) $\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_i + \sum_{k=1}^{n} b_k$

(3) $\sum_{k=1}^{n} (a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(4) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$, $1 \le m \le n$

Example

Evaluate each of the following sigma notations:

(a) $\sum_{k=1}^{3} 4k$ (b) $\sum_{k=1}^{5} (3k\text{-}2)$ (c) $\sum_{k=1}^{6} 2^{k-1}$

Solution

(a) $\sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24$

Using the above property (1), you can also calculate,

$$\sum_{k=1}^{3} 4k = 4\sum_{k=1}^{3} k = 4(1+2+3) = 4(6) = 24$$

(b) $\sum_{k=1}^{5} (3k-2) = 1 + 4 + 7 + 10 + 13 = 35$

Using property (1) and (3), you can also calculate,

$$\begin{aligned}\sum_{k=1}^{5} (3k-2) &= 3\sum_{k=1}^{5} k - \sum_{k=1}^{5} 2 = 3(1+2+3+4+5) - (2+2+2+2+2) \\ &= 3(1+2+3+4+5) - (2+2+2+2+2) \\ &= 3(15) - 2(5) \\ &= 35\end{aligned}$$

(c) $\sum_{k=1}^{6} 2^{k-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$$\begin{aligned}&= 1 + 2 + 4 + 8 + 16 + 32 \\ &= 63\end{aligned}$$

Using the above property (4), you can also calculate,

$$\begin{aligned}\sum_{k=1}^{6} 2^{k-1} &= \sum_{k=1}^{3} 2^{k-1} + \sum_{k=4}^{6} 2^{k-1} \\ &= (2^0 + 2^1 + 2^2) + (2^3 + 2^4 + 2^5) \\ &= 63\end{aligned}$$

Exercise 1.12

Evaluate each of the following sigma notations.

a. $\sum_{k=1}^{4} 5k$ b. $\sum_{k=1}^{5}(4k-1)$ c. $\sum_{k=3}^{6}(k^2-4)$

d. $\sum_{k=2}^{5} 3$ e. $\sum_{k=1}^{8}(k^3+2k^2-3k+5)$

1.3.2 Sum of Arithmetic Sequences

Activity 1.5

Find the sum of the first ten terms of the sequence $5, 15, 25, 35, \ldots$

HISTORICAL NOTE

Carl Friedrich Gauss (1777-1855)

A teacher of Gauss, at his elementary school, asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{r} 1+\ \ 2+\ 3+\ldots+100 \\ 100+99+98+\ldots+\ \ 1 \\ \hline 101+101+101+\ldots+101 \end{array}$$

$$\frac{100\times 101}{2}=5050.$$

To find the sum of the first 100 natural numbers, Gauss worked as follows. Writing the sum forward and backward then adding them together yields:

$$S_{100}=1+2+3+\ldots+98+99+100$$
$$S_{100}=100+99+98+\ldots+3+2+1$$
$$2S_{100}=101+101+101+\ldots+101+101+101$$
$$2S_{100}=100\times 101$$

Therefore, $S_{100} = \frac{1}{2} \times 100 \times 101 = 5050$

The sum of the first n natural numbers can also be calculated as follows:

$$S_n = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$$
$$S_n = n + (n-1) + (n-2) + ...3 + 2 + 1$$
$$2S_n = (n+1) + (n+1) + ... + (n+1) + (n+1)$$

Therefore, $2S_n = n(n+1)$

The sum of the first n consecutive natural numbers is $S_n = \frac{n}{2}(n+1)$.

Example

Find the sum of the first a. 40 natural numbers b. 150 natural numbers

Solution

a) Using the formula $S_n = \frac{n}{2}(n+1)$, $S_{40} = \frac{40}{2}(40+1) = 20(41) = 820$

b) Using the formula $S_n = \frac{n}{2}(n+1)$, $S_{150} = \frac{150}{2}(150+1) = 75(151) = 11,325$

Exercise 1.13

1. Find the sum of the first
   a. 30 natural numbers.
   b. 99 natural numbers.
   c. 200 natural numbers.
2. If the sum of the first n natural numbers is 3240, what is the value of n ?

Derivation of sum of arithmetic sequence

Let $\{A_n\}_{n=1}^{\infty}$ be an arithmetic sequence.

$$S_n = A_1 + A_2 + A_3 + ... + A_n \text{, where } A_n = A_1 + (n-1)d$$
$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + (A_1 + 3d) + ... + (A_1 + (n-1)d)$$

By collecting all the A_1 terms (there are n of them) we get,

$$S_n = nA_1 + [d + 2d + 3d + ... + (n-1)d]$$

Now factoring out d from within the brackets,

$$S_n = nA_1 + d[1 + 2 + 3 + ... + (n-1)]$$

Inside the brackets you have the sum of the first $(n-1)$ positive integers. Thus by using the formula, $S_n = \frac{n}{2}(n+1)$,

$$S_n = nA_1 + d\left(\frac{n-1}{2}\right)n = \frac{2nA_1 + n(n-1)d}{2} = \frac{n}{2}[2A_1 + (n-1)d].$$

This formula can be written as:

$$S_n = \frac{n}{2}[A_1 + \{A_1 + (n-1)d\}]$$

$$= \frac{n}{2}(A_1 + A_n)$$

Hence, the following theorem is derived:

Theorem 1.3

The sum S_n of the first n terms of an arithmetic sequence with first term A_1 and common difference d is $S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d]$.

This formula can also be written as:

$S_n = n\left(\frac{A_1 + A_n}{2}\right)$, where A_n is the n^{th} term.

This alternative formula is useful when the first and the last terms are known.

Example 1

Find the sum of the first 35 terms of the sequence whose general term is $A_n = 5n$.

Solution

From the general term, we get $A_1 = 5, A_2 = 10, A_3 = 15, \ldots,$ this shows that the given sequence is an arithmetic sequence. So, $A_{35} = 5(35) = 175.$

Since we can easily identify the first and the 35^{th} terms, we use the formula

$$S_n = \frac{n}{2}(A_1 + A_n) = n\left(\frac{A_1 + A_n}{2}\right)$$

Thus, substituting $A_1 = 5$, and $A_{35} = 175$

$$S_{35} = \frac{35}{2}(5+175) = 35\left(\frac{5+175}{2}\right) = 35(90) = 3{,}150$$

Example 2

If the 1st term of arithmetic sequence is 4, common difference is 5, then find the sum of the first 40 terms.

Solution

Given $A_1 = 4, \quad d = 5,$

$$S_n = \frac{n}{2}\left[2A_1 + (n-1)d\right] \Rightarrow S_{40} = \frac{40}{2}\left[2(4) + (40-1)(5)\right] = 20(8+195) = 20(203) = 4060.$$

Exercise 1.14

1. Find the partial sum of the following arithmetic sequences:
   a. $A_1 = 2,$ and last term $A_{10} = 21.$
   b. $A_1 = 40,$ and last term $A_{26} = 0.$
2. Find the sum of the following arithmetic sequences:
   a. $A_1 = 2, \ d = 3, \ n = 10$ b. $A_1 = 30, \ d = -5, \ n = 12$

Further on sum of arithmetic sequence

Example 1

Find the sum S_7 of the arithmetic sequence whose 4^{th} term is 2 and 7^{th} term is 17.

Solution

Applying the formula $A_n = A_1 + (n-1)d$

$$\begin{cases} A_4 = A_1 + 3d = 2 \\ A_7 = A_1 + 6d = 17 \end{cases}$$

Subtracting the first equation from the second equation,

$$3d = 17 - 2$$
$$d = 5$$

Thus,

$$A_1 = A_4 - 3d$$
$$A_1 = 2 - 3\times 5 = -13$$
$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$
$$S_7 = 7\left(\frac{-13+17}{2}\right) = 7(2) = 14.$$

Example 2

For a given arithmetic sequence the sum $S_{10} = 165$ and $A_1 = 3$, find A_{10}.

Solution

Since the first term and the sum are given, applying the formula gives:

$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$

$$S_{10} = 165, A_1 = 3, A_{10} = ?$$

$$S_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$$

$$165 = \frac{10}{2}(3 + A_{10})$$

$$\frac{165}{5} = 3 + A_{10}$$

$$A_{10} = 33 - 3 = 30.$$

Example 3

Find the sum of integers from 1 to 100 that are divisible by 10.

Solution

The number of such integer is $\frac{100}{10} = 10$

The 1^{st} term is 10, and the last term is 100. Then, $S_n = 10\left(\frac{10+100}{2}\right) = 550.$

Exercise 1.15

1. Find S_5 of the arithmetic sequence whose 3^{rd} term is 5 and 5^{th} term is 11.
2. Given the sum of an arithmetic sequence is $S_8 = 120$ and $A_1 = 1$, find A_8 and A_n.
3. Find the sum of the integers from 1 to 100 that are divisible by 2 or 5.
4. Find the sum of odd integers from 1 to 2001.

1.3.2 Sum of Geometric Sequences

Activity 1.6

1. Find the sum of the following geometric sequences:

   a. $\{2, 4, 8, 16, 32\}$ b. $\left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}\right\}$

2. Find the sum of the first 5 terms of the sequence $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

In order to answer such types of problems, consider the following:

Let $\{G_n\}_{n=1}^{\infty}$ be a geometric sequence, then its associated geometric sum, S_n is:

$$S_n = G_1 + G_2 + G_3 + \ldots + G_{n-1} + G_n, \text{ where } G_n = r^{n-1}G_1$$
$$S_n = G_1 + rG_1 + r^2G_1 + \ldots + r^{n-2}G_1 + r^{n-1}G_1$$

Factorizing out G_1,

$$S_n = G_1(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}) \quad (1)$$

Multiplying both sides of equation (1) by r

$$rS_n = G_1(r + r^2 + r^3 \ldots + r^{n-1} + r^n) \quad (2)$$

Subtracting the equation (2) from equation (1),

$$S_n - rS_n = G_1(1 + r + r^2 + r^3 + \ldots + r^{n-2} + r^{n-1}) - G_1(r + r^2 + r^3 + \ldots + r^{n-1} + r^n)$$
$$(1-r)s_n = G_1(1-r^n),$$
$$S_n = \frac{G_1(1-r^n)}{1-r} \text{ for } r \neq 1$$

Thus, the following theorem is inferred:

Theorem 1.4

Let $\{G_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio r. Then the sum of the first n terms S_n is given by, $S_n = \begin{cases} nG_1, & \text{if } r = 1 \\ G_1\dfrac{(1-r^n)}{1-r} = G_1\dfrac{(r^n-1)}{r-1}, & \text{if } r \neq 1. \end{cases}$

Example

Find the sum of the sequences in activity 1.6 using this theorem, and confirm the results.

Solution

a. $r = 2$, $G_1 = 2$ and $n = 5$

$$S_5 = 2\left(\frac{2^5 - 1}{2 - 1}\right) = 62$$

b. $\left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}\right\}$

$G_1 = 1$, $r = \frac{2}{3}$ and $n = 4$

$$S_4 = 1\left(\frac{\left(\frac{2}{3}\right)^4 - 1}{\frac{2}{3} - 1}\right) = \frac{65}{27}$$

Exercise 1.16

1. Find the sum of the from 1^{st} to n^{th} of the following geometric sequences:

   a. $G_1 = 3, \quad r = 2$ b. $G_1 = 1, \quad r = \frac{1}{2}$

2. Given the geometric sequence: 1, 3, 9, 27,.... find S_n.

Further on the sum of geometric sequences

Example

The sum of the first three terms of a geometric sequence is 7, and the sum from 4^{th} to 6^{th} terms is 56. Find the first term and the common ratio.

Solution

Let the first term be G_1 and common ratio r. Then,

$$G_1 + G_1r + G_1r^2 = 7 \qquad (1)$$

$$G_1r^3 + G_1r^4 + G_1r^5 = 56 \qquad (2)$$

From (2),

$$r^3(G_1 + G_1 r + G_1 r^2) = 56$$

Substituting (1),

$$7r^3 = 56$$
$$r^3 = 8$$
$$r = 2$$

Then,

$$G_1 + 2G_1 + 4G_1 = 7$$
$$7G_1 = 7$$
$$G_1 = 1$$

Therefore, the common ratio is 2, and the 1^{st} term is 1.

Exercise 1.17

1. The sum of the first three terms of a geometric sequence is 9, and the sum from the 4^{th} to 6^{th} term is -18. Find the first term and common ratio.
2. How many terms of the sequence: $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ are needed to give the sum $\frac{3069}{512}$?
3. Find the sum to indicated number of terms in each of the geometric sequence in questions *a* to *d*:

   a. $0.15, 0.015, 0.0015, \ldots$ *n* terms.

   b. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ *n* terms.

   b. $1, -a, a^2, -a^3, \ldots$ *n* terms (if $a \neq -1$).

   d. $x^3, x^5, x^7, \ldots$ *n* terms (if $x \neq \pm 1$).

1.4 Infinite Series

Activity 1.7

Supposing that a tree grows by half its height in a year. It then grows half of the amount of the previous year. What would be the height of the tree be if it continues to grow at the same rate?

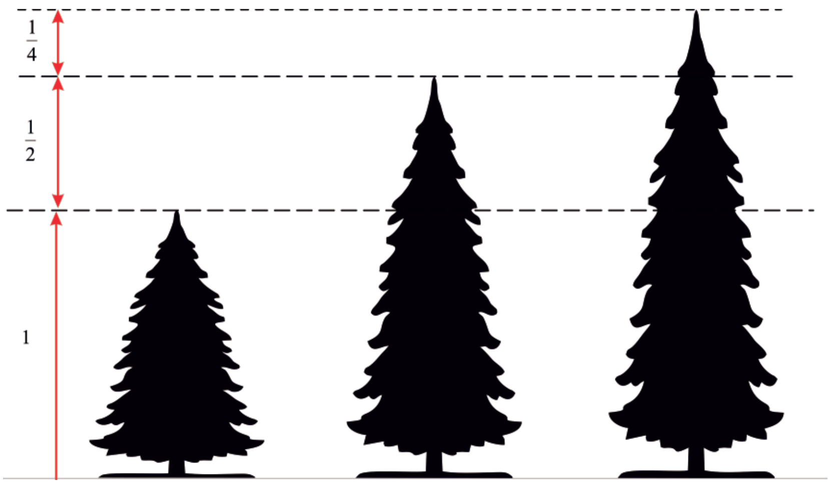


Figure 1.5

Suppose we have the sequence: $a_1,\ a_2,\ a_3,\ \dots,\ a_n,\dots$

An infinite sum of the form: $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called an infinite series and using summation, we can write as: $a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$

1.4.1. Divergence and Convergence of Infinite Sequence

Activity 1.8

For each of the sequences i – v:

a. Write the formula for the n^{th} term.

b. Find the sum where the sequence converges.

i. $1,\ 2,\ 3,\ 4,\dots$

ii. $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \dots$

iii. $-1,\ 1,\ -1,\ 1,\dots$

iv. $1,\ 1.5,\ 2.25,\ 3.375,\dots$

v. $0.12,\ 0.0012,\ 0.000012,\ 0.00000012,\dots$

For an infinite sequence, there are cases of convergence and divergence, when n approaches to infinity, as follows:

Convergence

a. Sequence a_n converges to α: $n \to \infty,\ a_n \to \alpha$

Divergence

b. Sequence a_n diverges to positive infinity: $n \to \infty,\ a_n \to \infty$.

c. Sequence a_n diverges to negative infinity: $n \to \infty,\ a_n \to -\infty$.

d. Sequence a_n vibrates: a_n has no limit. i.e. the value oscillate or vibrate back and forth between numbers.

Divergence/convergence of infinite geometric sequence

Considering infinite geometric sequence $G_n = r^n$, where $G_1 = r$, and the common ratio is r, there are cases of convergence and divergence, when n approaches to infinity, as follows:

i) $r > 1$, When $n \to \infty$, then $G_n \to \infty$ (diverge)

ii) $r = 1$, When $n \to \infty$, then $G_n \to 1$ (converge)

iii) $|r| < 1$, When $n \to \infty$, then $G_n \to 0$ (converge)

iv) $r \le -1$, When $n \to \infty$, then G_n vibrates (no limit, diverge).

Example

Find whether the given geometric sequences diverge, converge or vibrate as n approaches to infinity.

a. $G_n = (\sqrt{2})^n$ b. $G_n = \left(\frac{2}{3}\right)^n$ c. $G_n = (-3)^n$

Solution

a. $G_n = (\sqrt{2})^n$

As $\sqrt{2} > 1$, when $n \to \infty$, then $G_n \to \infty$, it diverges.

b. $G_n = \left(\frac{2}{3}\right)^n$

As $\left|\frac{2}{3}\right| < 1$, when $n \to \infty$, then $G_n \to 0$, it diverges.

c. $G_n = (-3)^n$

As $-3 \le -1$, when $n \to \infty$, then G_n vibrates

Exercise 1.18

Find whether the given geometric sequences diverge, converge or vibrates as n approaches infinity.

a. $G_n = (\sqrt{3})^n$ b. $G_n = \left(\frac{3}{4}\right)^n$ c. $G_n = \left(-\frac{1}{2}\right)^n$

Infinite series

Definition 1.5

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and S_n be the n[th] partial sum such that, as n gets larger and larger, S_n tends to s, where s is a real number, then we say the infinite series $\sum_{n=1}^{\infty} a_n$ converges and is written as $\sum_{n=1}^{\infty} a_n = s$.

However, if such an s does not exist or is infinite, we say the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Let us consider the geometric sequence $1, \frac{2}{3}, \frac{4}{9}, \ldots$ where $G_1 = 1$ and $r = \frac{2}{3}$. We have partial sum,

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

Let us study the behavior of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger, approaching to infinity.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes increasingly large, $\left(\frac{2}{3}\right)^n$ becomes increasingly small. In other words, as $n \to \infty$, $\left(\frac{2}{3}\right)^n \to 0$. Consequently, we find set on the line below the sum of infinitely many terms of the sequence above is $S = 3$.

Thus, for an infinite geometric sequence $G_1, G_1r, G_1r^2, \ldots,$ if numerical value of the common ratio r is between -1 and 1, then $S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1r^n}{1-r}$.

In this case, $r^n \to 0$ as $n \to \infty$ since $|r| < 1$ and then $\frac{G_1r^n}{1-r} \to 0$. Therefore, $S_n \to \frac{G_1}{1-r}$ as $n \to \infty$. Symbolically, the sum to infinity of an infinite geometric series is denoted by S_∞. Thus, we have $S_\infty = \frac{G_1}{1-r}$.

Note

Recall that if $\sum_{n=1}^{\infty} G_1 r^{n-1} = G_1 + G_1 r + G_1 r^2 + \ldots$ is a geometric series with first term G_1 and common ratio r, then $S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r}$.

(i) If $|r|<1$, as $n \to \infty$, $r^n \to 0$ so that $S_n = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r} \to \frac{G_1}{1-r}, S_\infty = \frac{G_1}{1-r}$.

(ii) If $|r| \geq 1$, then the series diverges or vibrates as follows.

a. If $r = 1$, then $\sum_{k=1}^{n} G_k = nG_1$, when $n \to \infty$, $\sum_{k=1}^{\infty} G_k = \infty \ (G_1 > 0)$

$$\sum_{k=1}^{\infty} G_k = -\infty \ (G_1 < 0)$$

b. If $r \neq 1$, then $S_n = \sum_{k=1}^{\infty} G_k = \frac{G_1(1-r^n)}{1-r}$

Then, If $r \leq -1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k$ vibrates,

If $r > 1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k = \infty \ (G_1 > 0)$

$$\sum_{k=1}^{\infty} G_k = -\infty \ (G_1 < 0)$$

Example 1

Determine whether the series $\sum_{n=1}^{\infty} 3^n$ converges or diverges.

Solution

The series $\sum_{n=1}^{\infty} 3^n = 3 + 9 + 27 + \ldots + \ldots$ is a geometric series with first term $G_1 = 3$ and common ratio $r = 3$.

Hence, the partial sum is given by $S_n = \frac{G_1(1-r^n)}{1-r}$

Substituting the values, we obtain,

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{3[1-3^n]}{1-3} = -\frac{3}{2}[1-3^n] = -\frac{3}{2}+\frac{3}{2}3^n.$$

Thus, as $n \to \infty$, $S_n \to \infty$. Therefore, the series diverges.

Example 2

Find the sum of the geometric series $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ldots$

Solution

The geometric series has the first term $G_1 = 5$ and the common ratio $r = \frac{-\frac{10}{3}}{5} = -\frac{2}{3}$.

Since

$|r| = \left|\frac{-2}{3}\right| = \frac{2}{3} < 1,$

$$S_\infty = 5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ldots = \frac{G_1}{1-r} = \frac{5}{1-\left(\frac{-2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3.$$

then the series converges and its sum is given by

Exercise 1.19

Find the sum for each of the following, if it exists, assuming the patterns continue as in the first few terms.

a. $3+1+\frac{1}{3}+\frac{1}{9}+\ldots$

b. $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ldots$

c. $\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\ldots$

d. $\frac{1}{5}+\frac{-1}{10}+\frac{1}{20}+\frac{-1}{40}+\ldots$

e. $7+\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\ldots$

Further on infinite series

Example 1

Find the sum $\sum_{k=1}^{\infty} 5^{3-k}$

Solution

$$5^{3-k} = 5^3 \times 5^{-k} = 5^3\left(\frac{1}{5}\right)^k = 5^2\left(\frac{1}{5}\right)^{k-1}$$

It is a geometric series whose first term $G_1 = 25$ and the common ratio $r = \frac{1}{5}$.

Therefore, the sum becomes:

$$S_\infty = \sum_{k=1}^{\infty} 5^{3-k} = 5^2 + 5^1 + 5^0 + 5^{-1} + \ldots = \frac{G_1}{1-r} = \frac{25}{1-\frac{1}{5}} = \frac{125}{4}$$

Example 2

Find the sum $\sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{2}{3}\right)^k$

Solution

$$(-1)^{k+1}\left(\frac{2}{3}\right)^k = (-1)^k(-1)\left(\frac{2}{3}\right)^k = (-1)\left(-\frac{2}{3}\right)^k = -\left(-\frac{2}{3}\right)^k$$

It is a geometric series whose first term $G_1 = \frac{2}{3}$ and the common ratio $r = -\frac{2}{3}$.

Therefore, the sum becomes

$$S_\infty = \sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{2}{3}\right)^k = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 + \ldots = \frac{G_1}{1-r} = \frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{2}{5}$$

Exercise 1.20

Find the sums for each of these geometric series

a. $\sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k-1}$ b. $\sum_{k=1}^{\infty} 2^{1-k}$ c. $\sum_{k=1}^{\infty} 5^{k-3}$ d. $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-2}$

1.4.2. Recurring Decimals

Recurring or repeating decimals are rational numbers (fractions) whose representations as a decimal contain a pattern of digit that repeats indefinitely after decimal point. The decimal that start their recurring cycle immediately after the decimal point are called **purely recurring decimals.**

Purely recurring decimals are converted to an irreducible fraction whose prime factors in the denominator can only be the prime numbers other than 2 or 5, i.e. the prime numbers from the sequence $\{3,7,11,13,17,19,...\}$. The decimals that have some extra digits before the repeating the sequence of digits are called the **mixed recurring decimals**.

The repeating sequence may consist of just one digit or of any finite number of digits. The number of digits in the repeating pattern is called **the period.**

Mixed requiring decimals are converted to an irreducible fraction whose denominator is a product of 2's and/or 5's besides the prime numbers from the sequence $\{3,7,11,13,17,19,...\}$.

All recurring decimals are infinite decimals.

Converting purely recurring decimals to fraction

Example 1

Convert the recurring decimal $0.\dot{3}$ to a fraction.

Solution

We can write the given decimal as the sum of the infinite converging geometric series as follows:

$$0.\dot{3} = 0.333333... = 0.3 + 0.03 + 0.003 + ...$$
$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

This is the infinite geometric series, where $G_1 = \frac{3}{10}$ and $r = \frac{\frac{3}{100}}{\frac{3}{10}} = \frac{1}{10}$

Since, $|r| < 1$ then $S_\infty = \frac{G_1}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$

Example 2

Convert the recurring decimal $0.\dot{4}\dot{7}$ to a fraction.

Solution

We can write the given recurring decimal as the sum of the infinite converging geometric series as follows:

$$0.\dot{4}\dot{7} = 0.474747... = 0.47 + 0.0047 + 0.000047 + ...$$
$$= \frac{47}{100} + \frac{47}{10000} + \frac{47}{1000000} + ...$$

Since, $r = \frac{G_n}{G_{n-1}} = \frac{1}{100}$, $|r| < 1$ then $S_\infty = \frac{G_1}{1-r}$,

Therefore, by substituting $G_1 = \frac{47}{100}$ and $r = \frac{1}{100}$ in to the formula:

$$0.\dot{4}\dot{7} = \frac{G_1}{1-r} = \frac{\frac{47}{100}}{1-\frac{1}{100}} = \frac{\frac{47}{100}}{\frac{99}{100}} = \frac{47}{99}$$

Notice that when converting a recurring decimal that is less than one to a fraction, we write the repeating digits to the numerator and in the denominator of the equivalent fraction write as many 9's as is the number of digits in the repeating pattern.

Exercise 1.21

Convert the following recurring decimals to fractions.

a. $0.\dot{4}$ b. $3.\dot{7}$ c. $0.\dot{5}\dot{6}$

1.5 Applications of Sequence and Series in Daily Life

This section is devoted to the application of arithmetic and geometric progressions or geometric series that are associated with real-life situations.

Activity 1.9

1. A man accepts a position with an initial salary of 5200 ETB per month. It is understood that he will receive an automatic increase of 320 ETB in the very next month and each month thereafter.
   a. Find his salary for the tenth month.
   b. What are his total earnings during the first year?
2. A carpenter was hired to build 192 window frames. The first day he/she made five frames and each day, thereafter he/she made two more frames than he /she made the day before. How many days did it take him/her to finish the job?

Here are some examples followed by exercises. They illustrate some useful applications.

Example 1

Observe the pattern in Figure 1.5:

a. If the pattern continues, find the number of letters in the column containing the letter M.

b. If the total number of letters in the pattern is 361, which letter will be in the last column?

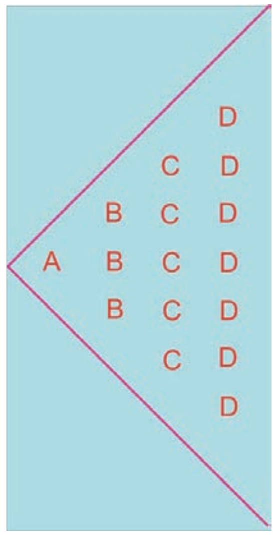


Figure 1.5

Solution

a. If you observe the structure of the letters in figure 1.5, it is of the form: $1, 3, 5, 7, 9, \ldots$

This is an arithmetic series with first term $A_1 = 1$ and common difference $d = 2$. Therefore, the n term of the arithmetic sequence is given by

$$A_n = A_1 + (n-1)d$$

For the letter M: $n = 13$;

$A_{13} = A_1 + (n-1)d = 1 + (13-1)(2) = 1 + (12)(2) = 25.$

b.

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$361 = \frac{n}{2}[2(1) + (n-1)(2)]$$

$$361 = n[1 + n - 1]$$

$$361 = n^2$$

$$n = \pm 19 \text{ (as } n > 0)$$

$$\therefore n = 19$$

So, the letter "S" will be in the last column.

Example 2

A theatre is filling up at a rate of 4 people in the first minute, 6 people in the second minute and 8 people in the third minute and so on. After 6 minutes, the theatre is half full. After how many minutes will the theatre be full?

Solution

The structure of the problem has an infinite arithmetic series of the form: $4+6+8+...$

So, the common difference of the problem is calculated as

$$d = A_2 - A_1 = A_3 - A_2 = 2$$

Therefore, the sum of arithmetic series whose first term $A_1 = 4$ and common difference $d = 2$, is written as

$$S_n = \frac{n}{2}[2A_1 + (n-1)d] \Rightarrow S_6 = \frac{6}{2}[2(4) + (6-1)(2)] = 3(18) = 54 \text{ (theatre half full).}$$

Therefore, the capacity of the theatre is $2 \times 54 = 108$

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$108 = \frac{n}{2}[2(4) + (n-1)(2)]$$

$$216 = n[8 + 2n - 2]$$

$$2n^2 + 6n - 216 = 0$$

$$n^2 + 3n - 108 = 0$$

$$(n+12)(n-9) = 0$$

$\therefore n = -12$ or $n = 9$, where n must be a positive integer.

Therefore, $n = 9$. It takes 9 minutes for the theatre to be full.

Example 3

A job applicant finds that a firm offers a starting annual salary of Birr 32,500 with a guaranteed raise of Birr 1,400 per year.

a. What would the annual salary be in the tenth year?

b. How much would be earned at the firm over the first 10 years ?

Solution

a. The annual salary at the firm forms the arithmetic sequence;

32,500, 33,900, 35,300, … with first term $A_1 = 32,500$ and common difference $d = 1,400$. Thus, $A_n = A_1 + (n-1)d$, substituting the values we obtain;

$A_{10} = 32,500 + (10-1)(1,400) =$ Birr 45,100

b. To determine the amount that would be earned over the first 10 years, we need to add the first 10 annual salaries;

$S_{10} = A_1 + A_2 + A_3 + ... + A_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$ (It is 10 times the average of the first and the last term.)

$$S_{10} = \frac{10}{2}(32,500 + 45,100) = \text{ Birr } 388,000$$

Therefore, a total of Birr 388,000 would be earned at the firm over the first 10 years.

Exercise 1.22

1. A person is scheduled to get a raise of Birr 250 every 6 months during his/her first 5 years on the job. If their starting salary is Birr 25,250 per year, what will his/her annual salary be at the end of the 3rd year?
2. Bontu begins a saving program in which she will save Birr 1,000 the first year. Each subsequent year, she will save 200 more than she did the previous year. How much will she save during the eighth year?

Example 4

A woman deposits Birr 3,500 in a bank account paying an annual interest at a rate of 6%. Show that whether the amounts she has in the account at the end of each year form a geometric sequence.

Solution

Let $G_1 = 3,500$. Then,

$$G_2 = G_1 + \frac{6}{100}G_1 = G_1(1+0.06) = G_1(1.06)$$

$$G_3 = G_2 + \frac{6}{100}G_2 = G_2(1+0.06) = G_1(1.06)(1.06) = G_1(1.06)^2$$

$$G_4 = G_3 + \frac{6}{100}G_3 = G_3(1+0.06) = G_1(1.06)^2(1.06) = G_1(1.06)^3$$

Continuing in this way, you get $G_n = G_1(1.06)^{n-1}$

Since the ratio of any two consecutive terms is a constant, which is 1.06, this sequence is a geometric sequence.

Example 5

Suppose a radioactive substance loses half of its mass per year. If we start with 100 grams of a radioactive substance, how much will be left after 10 years?

Solution

Let us record the amount of the radioactive substance left after each year starting with $G_0 = 100$. Note that each term is half of the previous term and hence,

$G_1 = \frac{1}{2}G_0 = 100\left(\frac{1}{2}\right)$ is the amount left at the end of year 1.

$G_2 = \frac{1}{2}G_1 = 100\left(\frac{1}{2}\right)^2$ is the amount left at the end of year 2.

$G_3 = \frac{1}{2}G_2 == 100\left(\frac{1}{2}\right)^3$ is the amount left at the end of year 3.

If you continue in this way, you see that the ratio of any two consecutive terms is a constant, which is $\frac{1}{2}$, and hence this sequence is a geometric sequence.

Therefore, after ten years, the amount of the substance left is given by:

$$G_{10} = G_0\left(\frac{1}{2}\right)^{10} = 100\left(\frac{1}{2}\right)^{10} = \frac{100}{1,024} = 0.09765625 \text{ g}.$$

Exercise 1.23

1. A certain item loses one-tenth of its value each year. If the item is worth Birr 28,000 today, how much will it worth 4 years from now?
2. A boat is now worth Birr 34,000 and loses 12% of its value each year. What will it worth after 5 years?
3. The population of a certain town is increasing at a rate of 2.5% per year. If the population is currently 100,000, what will the population be 10 years from now?
4. Sofia deposits Birr 3,500 in a bank account paying an annual interest rate of 6%.Find the amount she has at the end of the
   a. first year b. second year c. third year
   d. fourth year e. n^{th} year
   f. Do the amounts she has at the end of each year form a geometric sequence? Explain.

Example 6

A man was injured in an accident at work. He receives a disability grant of 4800 ETB in the first year. This grant increases by a fixed amount each year.

a. What is the annual rate of increase if he received a total of 143,500 ETB over 20 years?

b. His initial annual expenses are 2600 ETB, which increases at a rate of 400 ETB per year. After how many years will his expenses exceed his income?

Solution

a. Since the grant increases with a fixed amount each year, the problem is infinite arithmetic series and the sum is given as

$$4800+(4800+d)+(4800+2d)+...$$

This is an arithmetic series with first term is $A_1=4800$ and the common difference d. So, the sum of the arithmetic series is given by

$$S_n=\frac{n}{2}[2A_1+(n-1)d]$$

Therefore, the sum over 20 years is given by

$$\begin{aligned}S_{20}&=\frac{20}{2}[2(4800)+(20-1)d]=143,500\\143,500&=10[9600+19d]\\14,350&=9600+19d\\19d&=4750\\d&=250.\end{aligned}$$

Therefore, the annual increase if he received a total of 143,500 ETB over 20 years is 250 ETB.

b. The structure of the series is given by

$$2600+3000+3400+...$$

This is an arithmetic series whose first term $A_1=2600$ and common difference $d=400$.

So, the sum of the arithmetic series is given by

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$S_{expenses} = \frac{n}{2}[2(2600) + (n-1)(400)]$$

$$= \frac{n}{2}[5200 + 400n - 400] = \frac{n}{2}[4800 + 400n]$$

$$S_{income} = \frac{n}{2}[2(4800) + (n-1)(250)]$$

$$= \frac{n}{2}[9600 + 250n - 250] = \frac{n}{2}[9350 + 250n]$$

Let $S_{\text{expenses}} = S_{\text{income}}$

$$\frac{n}{2}[4800 + 400n] = \frac{n}{2}[9350 + 250n]$$

$$4800 + 400n = 9350 + 250n$$

$$150n = 4550$$

$$n = \frac{4550}{150} = 30.333...$$

Therefore, his expenses will exceed his income after 30 years.

Exercise 1.24

1. A job applicant finds that a firm A offers a starting salary of Birr 31,100 with a guaranteed raise of Birr 1,200 per year, whereas firm B offers a higher starting salary of Birr 35,100 but will guarantee a yearly raise of only Birr 900.
   a. What would the annual salary be in the 11^{th} year at firm A?
   b. What would the annual salary be in the 11^{th} year at firm B?
   c. Over the first 11 years, how much would be earned at firm A?
   d. Over the first 11 years, how much would be earned at firm B?
   e. Compare the amount earned in 11 years in firms A and B.
2. A contest offers a total of 18 prizes. The first prize is worth Birr 10,000, and each consecutive prize is worth Birr 500 less than the next higher prize. Find the value of the eighteenth prize and the total value of the prizes.

3. A contest offers 10 prizes with a total value of Birr 13,250. If the difference in value between consecutive prizes is Birr 250, what is the value of the first prize?

Example 7

A flower 110 cm high is planted in a garden. At the end of the first year, the flower is 120 cm tall. Thereafter the growth of the flower each year is half of its growth in the previous year. Show that the height of the flower will never exceed 130 cm. Draw a graph of the relationship between time and growth.

Solution

The annual growth of the flower: $10, 5, \frac{5}{2}, \frac{5}{4}, \ldots$ and the sum of the growth:

$$10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots$$

This is a geometric series with the first term $G_1 = 10$ and common ratio $r = \frac{1}{2}$.

$$S_\infty = \frac{G_1}{1-r} = \frac{10}{1-\frac{1}{2}} = 20.$$

Note: we may join the points on the graph because the growth is continuous.

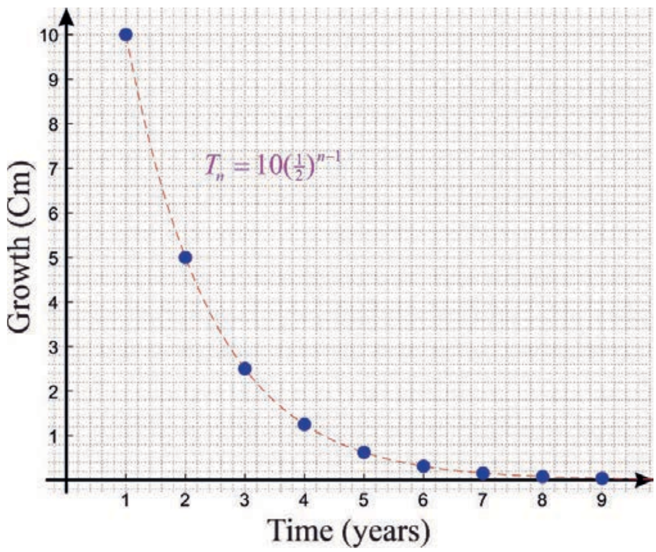


Figure 1.6

Therefore the growth of the flower is limited to 20 cm, and the maximum height of the shrub is therefore 110 cm + 20 cm = 130 cm.

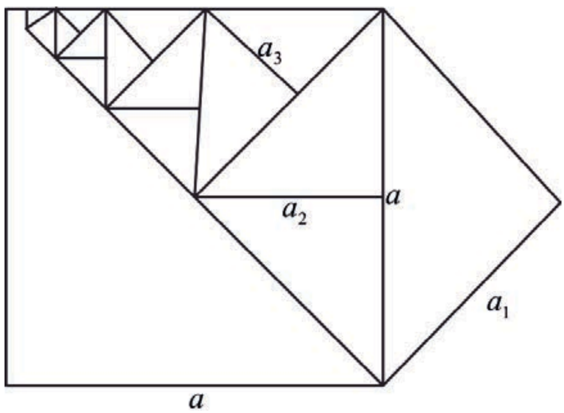
Example 8

Given a square with side length a. The side of the second square is half of its diagonal. The side of the third square is half of the diagonal of the second square and so on as shown in the figure below. Find the sum of the areas of all these squares.

Solution

Let the side of square be a_n and diagonal d_n.

Then,



$$a_0 = a$$

$$a_1 = \frac{1}{2}d_0 = \frac{1}{2}\sqrt{2}a = \frac{\sqrt{2}a}{2}$$

$$a_2 = \frac{1}{2}d_1 = \frac{1}{2}\sqrt{2}a_1 = \frac{\sqrt{2}}{2}\times\frac{\sqrt{2}a}{2}$$

$$a_3 = \frac{1}{2}d_2 = \frac{1}{2}\sqrt{2}a_2 = \frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}a}{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

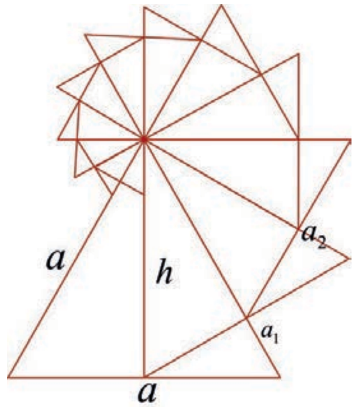
$$S_n = a_0{}^2 + a_1^2 + a_2^2 + a_3^2 + \ldots$$

$$= a^2 + \frac{1}{2}a^2 + \frac{1}{4}a^2 + \frac{1}{8}a^2 + \ldots$$

Since, $r = \dfrac{G_n}{G_{n-1}} = \dfrac{1}{2}$, $|r| < 1$, then $S_\infty = \dfrac{G_1}{1-r} = \dfrac{a^2}{1-\frac{1}{2}} = 2a^2$.

Exercise 1.25

1. Suppose a ball is dropped from a height of h m and always rebounds to r % of the height from which it falls. Show that the total vertical distance that could be covered by the ball is $h\left(\frac{r+1}{1-r}\right)$m. Assume that the ball will never stop bouncing.



2. Given an equilateral triangle with side length a, its height is the side of another equilateral triangle. The height of this triangle is then the side of the third equilateral triangle and so on, as shown in the diagram. Find the sum of the areas of all these triangles.

Summary

1. **Sequence**

- A sequence $\{a_n\}$ is a function whose domain is the set of positive integers or a subset of consecutive integers starting with 1.
- The sequence $\{a_1, a_2, a_3, ...\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n-1}^{\infty}$.
- A sequence that has a last term is called a **finite sequence**. Otherwise it is called **infinite sequence**.
- **Recursion formula** is a formula that relates the general term a_n of a sequence to one or more of the terms that come before it.

2. **Arithmetic and geometric progression**

- **An arithmetic sequence** is one in which the difference between consecutive terms is a constant, and this constant is called the common difference.
- If $\{A_n\}$ is an arithmetic progression with the first term A_1 and the common difference d, then the nth term is given by:

$$A_n = A_1 + (n-1)d.$$

- **A geometric progression** is one in which the ratio between consecutive terms is a constant, and this constant is called the common ratio.
- If $\{G_n\}$ is a geometric progression with the first term G_n and a common ratio r, then the nth term is given by: $G_n = G_1 r^{n-1}$.

3. **Partial sums:**

- The sum of the first n terms of the sequence $\{a_n\}_{n=1}^{\infty}$, denoted by S_n is called the partial sum of the sequence.
- The sum S_n of the first n terms of an arithmetic sequence with first term A_1, and common difference d is:

$$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d] \text{ or } S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[A_1 + A_n].$$

- In a geometric sequence, $\{G_n\}_{n=1}^{\infty}$, with common ratio r, the sum of the first n terms S_n is given by: $S_n = \begin{cases} nG_1, \text{ if } r = 1 \\ \dfrac{G_1(1-r^n)}{1-r}, \text{ if } r \neq 1 \end{cases}$

4. **Convergent series and divergent series**

- In a sequence if $\{a_n\}_{n=1}^{\infty}$, S_n is the n[th] partial sum such that, as $n \to \infty$, $S_n \to s$ where s is a real number, we say the infinite series $\sum_{n=1}^{\infty} a_n$ converges to s, otherwise the series diverges.

Review Exercise

1. List the first five terms of each of the sequences whose general terms are given below where n is a positive integer:

   a. $a_n = \frac{n^n}{n!}$ b. $a_n = (-1)^n + (-1)^n \sin(n\pi)$

   c. $a_n = Sgn(3-n)$

2. List the first five terms of each of the sequences whose general terms are given below where n is a positive integer:

   a. $a_n = 3^{\frac{1}{n}}(-1)^n$ b. $a_n = ne^{-2n}$ c. $a_n = \frac{(-2)^n + 6}{(n-1)!}$

   d. $a_n = (-1)^n - \frac{1}{n^2}$ e. $a_n = \cos\left(\frac{n\pi}{2}\right)$

3. Determine whether the sequences with the following general terms are arithmetic.

   a. $a_n = \frac{n^2 + 5n + 6}{n+2}$ b. $a_n = \sqrt{4n+1}$

4. Find the arithmetic mean between 1 and 9.

5. Find the 10^{th} and n^{th} terms of the geometric progression 5, 25, 125,…

6. Find three numbers in geometric progression such that their sum is $\frac{13}{3}$ and the sum of their squares is $\frac{91}{3}$.

7. Find the sum of

   a. $\sum_{k=1}^{8}(k^3 + 2k^2 - 3k + 5)$ b. $\sum_{k=2}^{5} 3$ c. $\sum_{k=1}^{5} \frac{1}{k^2 + 5k + 6}$

   d. $\sum_{m=1}^{10} \ln(\frac{m}{m+1})$

8. Let $\sum_{i=1}^{5} x_i = 37$, $\sum_{i=1}^{5} y_i = 127$, $\sum_{i=1}^{5} x_i^2 = 303$, $\sum_{i=1}^{5} y_i^2 = 50$ and $\sum_{i=1}^{5} x_i y_i = 105$.

Evaluate the following

a. $\sum_{i=1}^{5}(2x_i - 3y_i)$ b. $\sum_{i=1}^{5}(2x_i + 3y_i)$ c. $\sum_{i=1}^{5}(2x_i - 3y_i)^2$

d. $\left(\sum_{i=1}^{5} x_i\right)^2$ e. $\sum_{i=1}^{5}(2x_i - 5y_i + 3)$

9. Find the sum of all the natural numbers between 100 and 1000 which are multiples of 5.

10. In an arithmetic progression the 1st term is 2 and the sum of the 1st five terms is one-quarter of the next five terms. Show that the 20^{th} term is –112.

11. How many terms of the arithmetic progression $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum –25.

12. The sum of a certain number of terms of the arithmetic progression $25, 22, 19, \ldots$ is 116. Find the last term.

13. If the sum of n terms of an arithmetic progression is $(pn + qn^2)$, where p and q are constants, find the common difference.

14. If the n^{th} partial sum of an arithmetic sequence $\{A_n\}$ is $4n^2$, find A_n and A_{10}.

15. Convert this mixed recurring decimal $0.3\overline{17}$ to fraction.

16. Convert the mixed recurring decimal to fraction.

a. $0.3\overline{7}$ b. $3.23\overline{54}$

17. The first three terms of a convergent geometric series are: $x+1; x-1; 2x-5$.

a. Find the values of $x(x \neq 1 \text{ or } -1)$.

b. Find sum to infinity of the series.

18. Find $p: \sum_{k=1}^{\infty} 27p^k = \sum_{x=1}^{12}(24-3x)$.

19. Find the product $4 \times 4^{\frac{1}{2}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times \ldots \times 4^{\frac{1}{2^n}} \times \ldots$

20. If $\sum_{k=1}^{\infty} 5^{kt} = \frac{1}{24}$, find the values of t.

21. If the product $5^k \cdot 5^{k^2} \cdot 5^{k^3} \ldots = 5$, find k.

22. Evaluate $\sum_{k=1}^{11}(2+3^k)$.

23. The sum of the first three terms of a geometric progression is $\frac{39}{10}$ and their product is 1. Find the common ratio and the three terms.

24. How many terms of geometric progression $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

25. The sum of first three terms of a geometric progression is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the geometric progression.

26. Given a geometric progression with $a = 729$ and 7^{th} term 64, determine S_7.

27. If the 4^{th}, 10^{th} and 16^{th} terms of a geometric progression are x, y and z, respectively. Prove that x, y, z are in geometric progression.

28. Find the sum to n terms of the sequence $8, 88, 888, 8888 \ldots$.

29. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and $A, AR, AR^2, \ldots AR^{n-1}$ form a geometric progression, and find the common ratio.

30. If the p^{th}, q^{th} and r^{th} terms of a geometric progression are a, b and c, respectively. Prove that $a^{q-r}b^{r-p}c^{p-q} = 1$

31. If the first and the n^{th} term of a geometric progression are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$

32. If a, b, c and d are in geometric progression. Show that $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

33. Insert two numbers between 3 and 81 so that the resulting sequence is geometric progression.

34. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $(3+2\sqrt{2}):(3-2\sqrt{2})$.

35. If A and G are an arithmetic mean and geometric mean, respectively between two positive numbers,

prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

36. 150 workers were hired to finish a job in a certain number of days. 4 workers dropped out on second day, four more workers dropped out on third day and so on. It took eight more days to finish the work. Find the number of days in which the work was completed.

UNIT 2

INTRODUCTION TO CALCULUS

Unit Outcomes

By the end of this unit, you will be able to:

- Deduce rate of change of different quantities.
- Calculate rate of change of different quantities.
- Understand gradient of functions at a point.
- Analyze the geometrical and mechanical meaning of derivative.
- Find the derivative of simple functions using gradient method.
- Find area of a region bounded by a function and x-axis on a given interval.
- Understand definite integral.
- Value the real world contributions of having skills of derivatives and integrations.
- Apply the knowledge of integral calculus to solve real life mathematical problems.
- Apply the concept of derivatives to solve basic problems in business, economics, and road taxation.

Unit Contents

KEY Words

- Anti-derivative
- Area
- Critical number
- Definite integral
- Derivative
- Absolute maximum
- Partial sums
- Extreme values
- First derivative test
- Fundamental theorem
- Gradient
- Indefinite integral
- Partial fraction
- Differentiation
- Absolute minimum
- Rate of change
- Relative maximum
- Relative minimum
- Secant
- Tangent
- Volume

Introduction

In our daily life, we come across things that change according to some well recognizable rules. Many physical phenomena involve changing quantities such as the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the AC voltage of an electric signal, and so on.

Calculus is one of the components of mathematics that is concerned with change and motion; it deals with quantities that approach other quantities. Thus, the knowledge of calculus is essential, especially when dealing with quantities that change at a variable rate; you need to understand the concept of a derivative, which is the mathematical tool that is used to study the rates at which physical quantities change. The derivative is the exact rate at which one quantity changes with respect to another. Geometrically, the derivative is the slope of a curve at a point on the curve, defined as the slope of the tangent to the curve at the same point. The process of finding the derivative is called differentiation. This process is central to the branch of mathematics called differential calculus. In this unit, you are going to study the meaning and techniques of differentiation, applications of derivatives, and integration.

Historical Notes

Calculus, known in its early history as infinitesimal calculus, is a mathematical discipline focused on limits, continuity, derivatives, integrals, and infinite series. Isaac Newton and Gottfried Wilhelm Leibniz independently developed the theory of infinitesimal calculus in the late 17^{th} century. Newton was only 22 at the time.

2.1 Introduction to Derivatives

2.1.1 Understanding Rates of Change

Rate of change

Activity 2.1

1. Consider a square of length of side 4 cm.
   a. If the length of the side of the square increases by 1 cm, find the change in the perimeter of the square.
   b. If the length of the side of the square increases by 1 cm/s find the perimeter of the square when t = 1s, 2s, 3s.
   c. What is the time rate of change of the perimeter of the square when its length of side increases by 2 cm/s?
2. What does a rate of change mean?

The fundamental philosophical truth is that everything changes. In physics, the change in position is known as displacement, in economics, the price change is known as inflation, in business, the change in costs is sometimes known as a trend, in mathematics, the change in values of a function is known as the derivative. But, to understand the derivatives, which measure instantaneous change, we need to understand the concepts of change and average change over intervals.

Definition 2.1

A rate of change is a rate that describes how one quantity changes in relation to another quantity. The units on a rate of change are output units per input units.

For Example: Cost per minute, plants per hectare, kilometers per liter of benzene, salary per month, population per square kilometer, etc.

Rates of change can be positive or negative or even zero. When a quantity does not change over time, it has zero rates of change.

1. **A positive rate of change** is the case when the values of the two quantities (say x and y) increases at the same time and the graph slopes upward (See Figure 2.1).

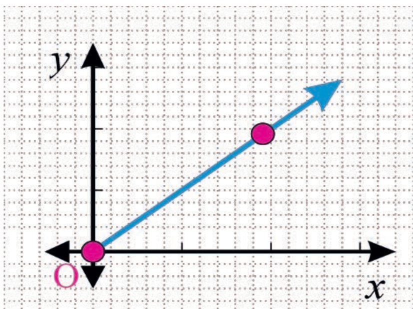


Figure 2.1

2. **Negative rate of change** is the case when the value of one quantity (say x) increases and the value of the other quantity (say y) decreases. The graph slopes downward (See Figure 2.2).

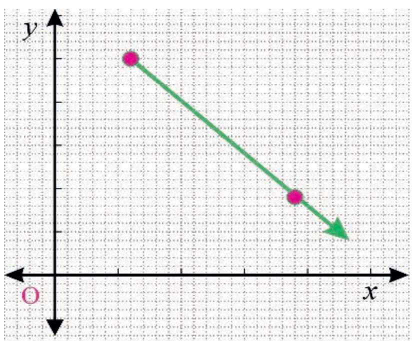


Figure 2.2

3. **Zero rate of change** is the case when the value of one quantity (say x) increases, the value of the other quantity (say y) remains constant. The graph is a horizontal line.

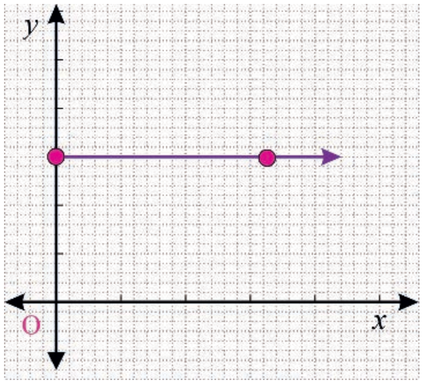


Figure 2.3

Example

Calculate the rate of change for the following pair of numbers.

a. $(1,2)$ and $(3,4)$ b. $(2,3)$ and $(3,1)$ c. $(0,2)$ and $(2,2)$ d. $(2,0)$ and $(2,2)$

Solution

a. $\text{Rate of change} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-2}{3-1} = \dfrac{2}{2} = 1.$

Thus, the rate of change is positive.

b. $\text{Rate of change} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-3}{3-2} = \dfrac{-2}{1} = -2.$

Thus, the rate of change is negative.

c. $\text{Rate of change} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-2}{2-0} = \dfrac{0}{2} = 0.$

Thus, the rate of change is zero.

d. $\text{Rate of change} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-0}{2-2} = \dfrac{2}{0}$ (undefined).

Thus, the rate of change is undefined.

Remark

A function is increasing where its rate of change is positive and decreasing where its rate of change is negative.

Exercise 2.1

Calculate the rate of change for the following pairs of numbers:

a. $(4,6)$ and $(5,9)$
b. $(4,7)$ and $(5,6)$
c. $(1,2)$ and $(3,-4)$
d. $(-2,4)$ and $(3,-6)$
e. $(0,9)$ and $(9,9)$
f. $(8,0)$ and $(8,8)$
g. $(3,-1)$ and $(1,-1)$

Note

The two kinds of rate of change are average rate of change and instantaneous rate of change.

Average rate of change

Definition 2.2

If $y = f(x)$ is a function, then the **average rate of change** of y with respect to x on the interval $[a,b]$ is given by the expression:

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}.$$

Example 1

Find the average rate of change of a function $f(x)=2x-9$ beteween $x_1 = 3$ and $x_2 = 6$.

Solution

Step 1: Write the formula for the average rate of change over the interval $x_1 \le x \le x_2$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Step 2: Solve for $f(x_1)$ and $f(x_2)$.

$$f(x_2)=f(6)=2(6)-9=12-9=3 \text{ and } f(x_1)=f(3)=2(3)-9=6-9=-3$$

Step 3: Substitute the known values into the formula and simplify.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{f(6)-f(3)}{6-3}=\frac{3-(-3)}{3}=2.$$

Example 2

Examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in meter) at 1-second intervals as in the following chart:

T(*s*)	0	1	2	3	4	5
S(*m*)	0	2	10	25	43	78

a. Find the average velocity during the time interval $2 \le t \le 4$.

b. Find the average velocity during the time interval $2 \le t \le 3$.

Solution

a. **Step 1:** Write the formula for the average rate of change over the interval $a \le x \le b$

$$\text{Average rate of change} = \frac{f(b)-f(a)}{b-a}$$

Step 2: Identify the two points corresponding to the interval

$$(a, f(a))=(2,10) \text{ and } (b, f(b))=(4,43)$$

Step 3: Substitute the values into the formula and simplify

$$\text{Average rate of change} = \frac{f(b)-f(a)}{b-a}=\frac{43-10}{4-2}=16.5\text{m/s}.$$

b. Similarly, Average velocity $= \dfrac{\text{distance travelled}}{\text{time elapsed}} = \dfrac{25-10}{3-2} = 15\text{m/s}.$

Example 3

The accompanying figure shows the graph of a function f. Find the average rate of change of the function f over the interval $0 \le x \le 9$:

Solution

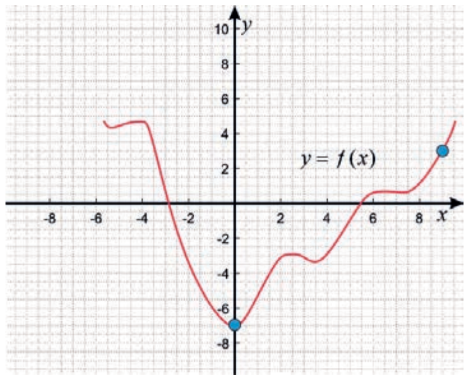


Figure 2.4

Step 1: Write the formula for the average rate of change over the interval $0 \le x \le 9$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}, \text{ where } x_1 = 0 \text{ and } x_2 = 9.$$

Step 2: Observe from the graph the value of $f(0)$ and $f(9)$. We can see from the graph that $f(0) = -7$ and $f(9) = 3$.

Step 3: Substitute the known values into the formula.

$$\text{Average rate of change} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(9)-f(0)}{9-0} = \frac{3-(-7)}{9} = \frac{10}{9}.$$

Note: The average rate of change is a measure of how much the function changed per unit, on average, over the interval. Geometrically, the average rate of change of a function $y = f(x)$ on the interval $[a,b]$ is the slope of the **secant line** through $(a, f(a))$ and $(b, f(b))$ See Figure 2.5.

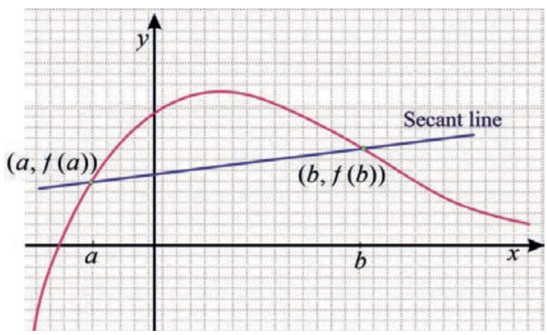


Figure 2.5

Exercise 2.2

1. Compute the average rate of change of
   a. $f(x)=3x-2$ over the interval $0 \le x \le 2$
   b. $f(x)=x^3-3x$ over the interval $1 \le x \le 6$
   c. $f(x)=3x-2$ on the interval $[1,5]$
   d. $f(x)=x^2+3x+1$ on the interval $[0,k]$
2. In the following table, the average cost in Birr of a gallon of gasoline for the years 2005–2013 is illustrated. Using the data in the table, find the average rate of change of the price of gasoline between 2007 and 2009.

x	2005	2006	2007	2008	2009	2010	2011	2012	2013
$c(x)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68	4.21

3. A candle has a starting length of 10 cm. Thirty minutes after lighting it, the length became 7 cm. Determine the rate of change in the length of the candle as it burns. Determine how long the candle takes to completely burn out.

Instantaneous Rate of Change

For many of the functions such as polynomial, trigonometric, exponential, logarithmic, and rational functions, it is possible to find their instantaneous rate of change for values of x in their domain.

Activity 2.2

Let $y=x^3$.

i. Find the average rate of change of the given function on the following intervals.

   a. $x=1$ and $x=1.1$ *b*. $x=1$ and $x=1.001$ *c*. $x=1$ and $x=1+\Delta x$

ii. Determine what happens as Δx approaches 0.

Definition 2.3

The instantaneous rate of change is the change at that particular moment.

Instantaneous rate of change $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, where Δx gets closer and closer to zero.

Note

Average rate of change is the rate of change over the entire period. The **instantaneous rate** of change is the change at a certain period of time.

Therefore, when h gets closer and closer to zero, the instantaneous rate of change of a function f at $x = x_0$ is $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$.

Example 1

Evaluate the instantaneous rate of change of $f(x) = 2x^2 + 9$ at $x = -3$.

Solution

The instantaneous rate of change of a function f at $x = x_0$ is

$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$, as h gets closer and closer to zero. For the given function $f(x) = 2x^2 + 9$ at $x = -3$, the instantaneous rate of change is

$$\text{Instantaneous rate of change} = \frac{f(-3+h) - f(-3)}{h}$$

$$= \frac{2(-3+h)^2 + 9 - (2(-3)^2 + 9)}{h} = \frac{2(h^2 - 6h + 9) + 9 - (18 + 9)}{h}$$

$$= \frac{2h^2 - 12h + 18 + 9 - (18 + 9)}{h} = \frac{2h^2 - 12h}{h}$$

$= 2h - 12$, as h gets closer and closer to zero

$= -12$.

Thus, the instantaneous rate of change is -12.

Example 2

Find the instantaneous rate of change of $f(x)=\sqrt{x}$ at $x=x_0>0$.

Solution

Recall that the instantaneous rate of change of a function $f(x)$ when $x=x_0$ is

$$\text{Instantaneous rate of change} = \frac{f(x_0+h)-f(x_0)}{h}$$

$$= \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h}, \text{ rationalizing the numerator}$$

$$= \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h}\left(\frac{\sqrt{x_0+h}+\sqrt{x_0}}{\sqrt{x_0+h}+\sqrt{x_0}}\right)$$

$$= \frac{h}{h(\sqrt{x_0+h}+\sqrt{x_0})}$$

$$= \frac{1}{\sqrt{x_0+h}+\sqrt{x_0}}, \text{ as } h \text{ gets closer and closer to zero}$$

$$= \frac{1}{2\sqrt{x_0}}.$$

Instantaneous rates of change have many real-world applications; for example, the velocity of a moving body at a particular time is the instantaneous rate of change of the displacement at that time. In the final example, we will consider a real-world problem and determine the instantaneous rate of change at a particular time for a polynomial function representing the biomass of a bacterial culture.

Example 3

The biomass of a bacterial culture in milligrams as a function of time in minutes is given by $f(t)=71t^3+63$. What is the instantaneous rate of growth of the culture when $t=2$ minutes?

Solution

In this example, we want to determine the instantaneous rate of change of a cubic function representing the biomass of a bacterial culture.

Recall that the instantaneous rate of change of a function $f(t)$ when $t = t_0$ is

$$
\begin{aligned}
\text{Instantineous rate of change} &= \frac{f(2+h)-f(2)}{h} \\
&= \frac{71(2+h)^3+63-(71\times 2^3+63)}{h} \\
&= \frac{71(h^3+6h^2+12h+8)-71\times 8}{h} \\
&= \frac{71h^3+426h^2+852h}{h} \\
&= 71h^2+426h+852, \text{ as } h \text{ gets closer and closer to zero} \\
&= 852.
\end{aligned}
$$

Since the rate of change is positive, it is equivalent to the rate of growth. Hence, the rate of growth of the biomass of a bacterial culture when $t = 2$ minutes is 852 mg/min.

The instantaneous rate of change is also related to the derivative, at an arbitrary point x. In particular, the instantaneous rate of change at $x = x_0$ of a function is the derivative of a function evaluated at $x = x_0$.

Exercise 2.3

1. Evaluate the instantaneous rate of change of $f(x) = 6x^2 - 3$ at $x = 1$.
2. Find the instantaneous rate of change of $f(x) = 2\sqrt{x}$ at $x = x_0 > 0$.
3. The biomass of a bacterial culture in milligrams as a function of time in minutes is given by $f(t) = 81t^3 + 90$. What is the instantaneous rate of growth of the culture when $t = 3$ minutes?

2.1.2 Gradient of Curves and Rate of Changes

Activity 2.3

1. What is a secant line to a curve? What is a tangent line to a curve?
2. What is slope of a line? How can we calculate the slope of a line?

Definition 2.4

a. **A tangent line** to a curve or a graph of a function $y = f(x)$ is a line that touches the curve exactly at one point, but does not cross the curve. The point where the tangent line touches the graph is said to be the **point of tangency.**

b. **Gradient (slope)** is a number that describes steepness and direction of a line. If slope is positive, then the line is increasing, which means it goes up as we move from left to right. If slope is negative then the line is decreasing, which means it goes down when we move from left to right. If the slope is zero, then the line is horizontal.

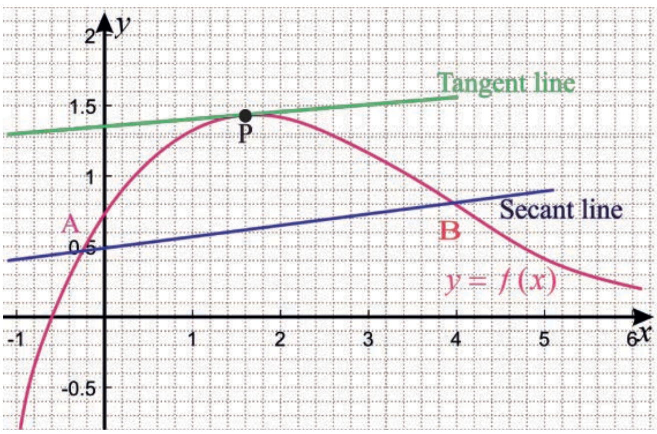


Figure 2.6

Note: If we are given two points $A(x_0, y_0)$ and $B(x_1, y_1)$ on the graph of $y = f(x)$, then using definition 2.2 and understanding that 'change' is the difference between the two values we get the general formula for the slope of a secant line through the two points, which is given as:

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

From this formula, we can see that the gradient of the secant line to the curve of a function is equal to the average rate of change.

The slope of a secant line = average rate of change.

Example 1

Find the slope of a line passing through the given two points.

a. $(-5,7)$ and $(0,0)$ **b.** $(3,-5)$ and $(-2,9)$

Solution

a. $\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{0-(7)}{0-(-5)} = \frac{-7}{5} = -\frac{7}{5}.$

b. $\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{(9-(-5))}{(-2-3)} = \frac{14}{-5} = -\frac{14}{5}.$

Example 2

Find the slope of a secant line to the graph of the function $f(x) = 2x^3 + 3x - 5$ over the interval [1, 3].

Solution

Given $f(x) = 2x^3 + 3x - 5$

Step 1: Write the formula for the slope of a secant line over the interval $1 \le x \le 3$.

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3-1}.$$

Step 2: Solve for $f(3)$ and $f(1)$.

$$f(3) = 2(3)^3 + 3(3) - 5 = 58 \text{ and } f(1) = 2(1)^3 + 3(1) - 5 = 0$$

Step 3: Substitute the known values into the formula and simplify.

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} = \frac{58 - 0}{2} = 29.$$

Therefore, the slope of a secant line to the graph of $f(x) = 2x^3 + 3x - 5$ with respect to x over the interval $[1, 3]$ is 29.

Exercise 2.4

1. Find the slope of a line passing through the two given points.

   a. $(-3, -5)$ and $(3, 11)$ b. $(4, -9)$ and $(5, -7)$

2. For each of the following functions, find the slope of a secant line to the graph of the functions over the given intervals.

   a. $f(x) = x^2 + 1; [-1, 3]$ b. $g(x) = x^3 - 4x + 6;\ [1, 2]$.

2.1.3 Gradient at a Point on a Curve

Activity 2.4

1. Let $f(x) = x^2$. Compute and simplify the algebraic expression

   a. $\dfrac{f(x+h) - f(x)}{h}$ b. $\dfrac{f(x) - f(x_0)}{x - x_0}$

2. Let $y = x^3$.

   i. Find the slope of the secant line to the curve of the function between

   a. $x = 1$ and $x = 1.1$ b. $x = 1$ and $x = 1.001$ c. $x = 1$ and $x = 1 + h$.

   ii. Determine what happens as h approaches 0.

Suppose you are given some curve, and a point P on the curve as shown below in figures 2.7 a and b.

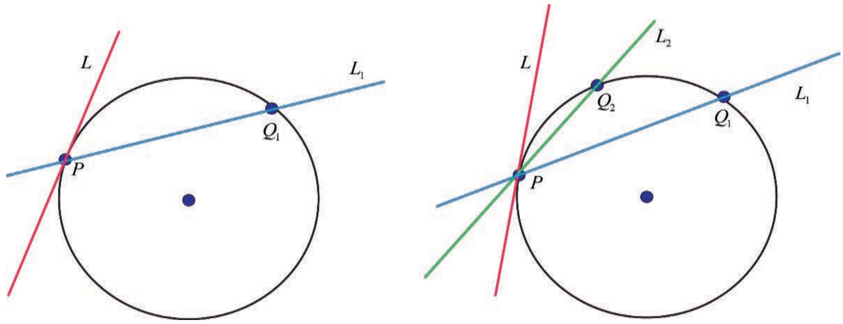


Figure 2.7a Figure 2.7b

- The line L passing through the point P is called a tangent line.
- Given another point Q_1 on the curve, let L_1 be the line passing through the points P and Q_1.
- L_1 is a secant line and is sometimes written as PQ_1.
- Pick a point Q_2 closer to P. Notice that the slope of the secant line L_2 is closer to the slope of L compared to the slope of L_1.
- As we pick points closer and closer to P, the slope of the associated secant lines will get closer and closer to the slope of the tangent line L.
- Let P (a, f (a)) be any point on the graph of a function f. Another point on the graph may be denoted by $Q\ (a+h, f\,(a+h))$, where h is the difference between the x-coordinates of Q and P (see Figure 2.8 a). By definition the slope m of the secant line through P and Q is

$$m_{PQ} = m_{\text{secant}} = \frac{f(a+h)-f(a)}{h}.$$

Note

m_{PQ} denotes the slope of the secant line through P and Q and m_{secant} represents the slope of the secant line to the graph of $f(x)$.

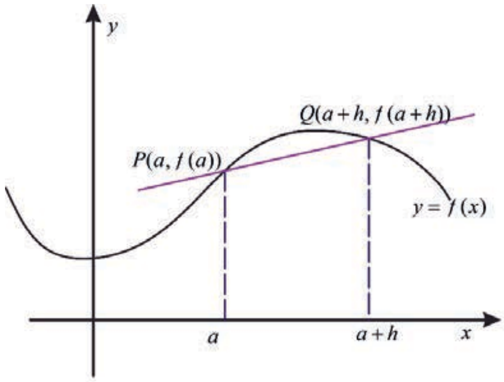


Figure 2.8 *a*

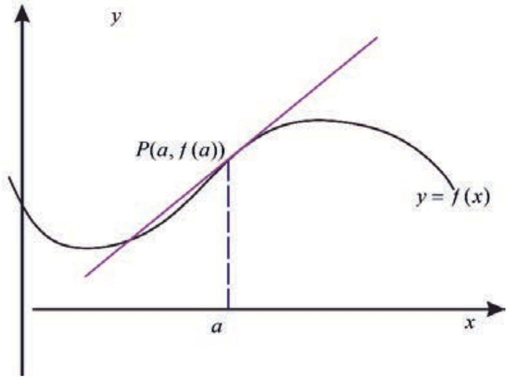


Figure 2.8 *b*

If *f* is a given function, then we can make *Q* approach *P* by letting *h* approach 0 (see figure 2.8*b*) and we get the slope of the tangent line to the curve. It is natural, therefore, to define *m* as follows:

Definition 2.5

If f is a function defined on an open interval containing a, then **as h gets closer and closer to zero**, the slope m of the tangent line to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by: $m = \dfrac{f(a+h) - f(a)}{h}$.

Note

1. If we allow the quantity h to get smaller and smaller, the point $(a+h, f(a+h))$ is getting closer to $(a, f(a))$, which in turn means that the slope of the secant line through $(a+h, f(a+h))$ is getting closer to the slope of the tangent line at $(a, f(a))$.

   Hence, the slope of the graph of a function at a point is the same as the slope of the tangent line at that point.
2. The slope of a tangent line represents the **instantaneous rate of change** of the function at that one point. That is:

   The slope of a tangent line = instantaneous rate of change.

Procedure to find gradient of a curve at a point

To find the gradient of a curve $y = f(x)$ at $x = a$ use the following procedure:

Step 1: Write the quotient difference $\dfrac{f(x) - f(a)}{x - a}$ of f at $x = a$.

Step 2: Compute the algebraic expression in step1 and simplify.

Step 3: Find the value of the result obtained in step 2 as the value of x approaches a.

The final result is the gradient of the tangent line to the curve $y = f(x)$ at $x=a$.

Example 1

Find the gradient (slope) of the functions $f(x) = x^2$ at $x = 2$.

Solution

$$\text{Slope} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2.$$

Therefore, the slope of the function at $x=2$ becomes

$$x + 2\big|_{x=2} = 2 + 2 = 4$$

Note: $x + 2\big|_{x=2}$ means value obtained by substituting $x = 2$ to $x + 2$.

Example 2

If $f(x) = x^2$ find the gradient of the tangent line to the graph of $f(x)$ at the point $P(a, a^2)$.

Solution

Given $y = f(x) = x^2$, using the above procedure, we have

i. $$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x-a)(x+a)}{x - a} = x + a$$

ii. As x approaches to a, the expression or the quantity $x + a$ approaches $2a$.

Therefore, the gradient of the tangent line to the curve $y = f(x) = x^2$ at the point $P(a, a^2)$ is $m_{\tan} = 2a$, where $m_{\tan}$ denotes the slope of the tangent line to the graph of the function at the specified point of tangency.

This means that the instantaneous rate of change of the function at the point P is $2a$, and that the gradient of the tangent to the graph of the curve $f(x) = x^2$ at any point is found by doubling the x-coordinate. For example, the gradient of the function $f(x) = x^2$ at the point $P(4,16)$ is equal to $m_{\tan} = 2(2) = 8$, at $P(-5,25)$ is $m_{\tan} = -10$, at the origin $P(0,0)$ is $m_{\tan} = 0$. See Figure 2.9.

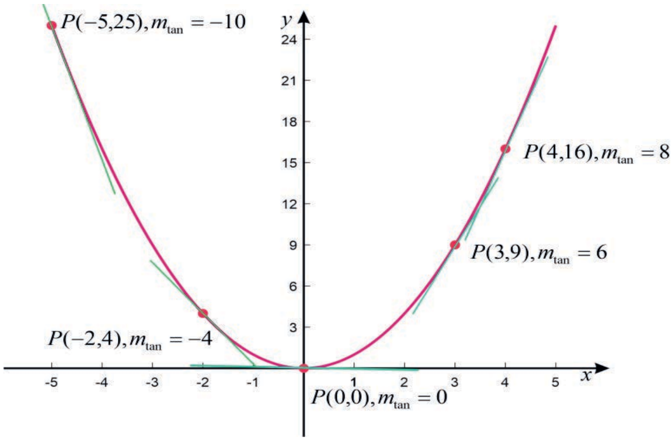


Figure 2.9

To clarify this idea condider the following table:

x	-5	-2	0	3	4
$f(x) = x^2$	25	4	0	9	16
Gradient (m)	-10	-4	0	6	8

Exercise 2.5

1. Find the gradients of the tangent lines to the graph of the following function at specified points.

   a. $f(x) = x^2$ at $x = -1$ and $x = 4$.

   b. $f(x) = 3x^2 - 5x + 4$ at $x = 2$ and $x = -\sqrt{2}$.

   c. $g(x) = \sqrt{x}$ and $x = 4$.

   d. $f(x) = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

2. If $f(x)=x^3$, find the gradient of the tangent line to the graph of $f(x)$ at the point $P(a,a^3)$.

2.1.4 Definition of Derivative

The slope concept usually pertains to straight lines. When a function is non-linear, its slope may vary from one point to the next. We must therefore introduce the notion of derivative which allows us to obtain the slope at all points of these non-linear functions.

2.1.4.1 Derivative of function at a point

Activity 2.5

Let $f(x)=\frac{1}{x^2}$. Then

a. Simplify $\frac{f(x)-f(1)}{x-1}$.

b. Simplify $\frac{f(x+h)-f(x)}{h}$.

Definition 2.6 (Derivatives of functions at a point)

The derivative of a function $f(x)$ at a number "a" in the domain of f, denoted by $f'(a)$, is the gradient of the tangent line to the graph of $y=f(x)$ at $(a, f(a))$. That is, as h gets closer and closer to zero from both directions, $\frac{f(a+h)-f(a)}{h}$ becomes closer to $f'(a)$.

Note

- $f'(a)$ is the slope of the line tangent to the graph of f at the point $(a, f(a))$.
- If $y=f'(x)$ is defined, then we say that f has a derivative at a or f is differentiable at a.

Example

Use definition 2.6, to find the derivatives of the following functions at the given value of a.

a. $f(x)=5x-3;\ a=-2$ b. $f(x)=x^2+3;\ a=5$ c. $f(x)=\sqrt{2x+4};\ a=6$

Solution

Using the definition, as h gets closer and closer to zero $f'(a)=\dfrac{f(a+h)-f(a)}{h}$,

and we have the following results:

a. $$f'(-2)=\frac{f(-2+h)-f(-2)}{h}$$
$$=\frac{[5(-2+h)-3]-(-2\times5-3)}{h}$$
$$=\frac{(-10+5h-3)-(-13)}{h}=\frac{5h}{h}=5.$$

b. $$f'(5)=\frac{f(5+h)-f(5)}{h}$$
$$=\frac{[(5+h)^2+3]-(5^2+3)}{h},$$
$$=\frac{25+10h+h^2+3-(25+3)}{h}$$
$$=\frac{h^2+10h}{h}$$
$$=h+10=10 \text{ as } h \text{ approaches to } 0$$

c. $$f'(6)=\frac{f(6+h)-f(6)}{h}=\frac{\sqrt{2(6+h)+4}-\sqrt{2(6)+4}}{h}$$
$$=\left(\frac{\sqrt{2h+16}-\sqrt{16}}{h}\right)\left(\frac{\sqrt{2h+16}+\sqrt{16}}{\sqrt{2h+16}+\sqrt{16}}\right), \text{ rationalization}$$
$$=\frac{2h+16-16}{h\left(\sqrt{2h+16}+\sqrt{16}\right)}, \text{ simplification}$$

$$= \frac{2}{\sqrt{2h+16}+\sqrt{16}} \quad \text{as } h \text{ approaches to 0,}$$

$$= \frac{2}{\sqrt{16}+\sqrt{16}} = \frac{1}{4}.$$

Exercise 2.6

Use definition 2.6, to find the derivatives of the following functions at the given value.

a. $f(x) = 9x - 7, \ a = 2$

b. $f(x) = x^2 - 1, \ a = 2$

c. $f(x) = x^3 + 2x, \ a = 3$

d. $f(x) = \sqrt{3x-6}, \ a = 3$

2.1.4.2 The derivative as a function

Activity 2.6

For each of the following functions, find the set of values of x_0 such that $f'(x_0)$ is defined and is unique.

a. $f(x) = x^2$

b. $f(x) = |x|$

c. $f(x) = \frac{1}{x}$

From Activity 2.6, you have observed that there are functions that are differentiable at all numbers in their domains and there are also functions that are not differentiable at some numbers in their domain. We are now ready to define derivatives.

Definition 2.7

The function f' whose domain consists of those values of x at which f is differentiable and whose value at any such number as h gets closer and closer to zero is given by $f'(x) = \frac{f(x+h) - f(x)}{h}$

Here, we represent the derivative of a function by a prime symbol. For example, writing $f'(x)$ represents the derivative of the function f evaluated at a point x.

Similarly, writing $(5x^2+2x+3)'$ indicates we are carrying out the derivative of the function $5x^2+2x+3$. The prime symbol disappears as soon as the derivative has been calculated. If we let $t=x+h$, as h gets closer and closer to zero, t approaches x and the definition above can be re-written as

$$f'(x)=\frac{f(t)-f(x)}{t-x} \text{ as } t \text{ gets very close to } x.$$

Note

The derivative is the exact rate at which one quantity changes with respect to another.

The different notations for the derivative

The following are some of the other notations for derivatives. If $y=f(x)$, then $f'(x)$ is denoted by either of the following notations: $\frac{dy}{dx}, \frac{d}{dx}f(x), Df(x), D_x f$

Example

Find the derivative of the following functions.

a. $f(x)=k, k$ be any constant   b. $f(x)=x^2$ c. $f(x)=\sqrt{x}$

Solution

Use the definition and let h get closer and closer to 0.

Now, from $f'(x)=\frac{f(x+h)-f(x)}{h}$, we obtain:

a. For $f(x)=k, k$ be any constant, the quotient $\frac{f(x+h)-f(x)}{h}=\frac{k-k}{h}=0.$

Thus, by definition 2.6 $f'(x)=0$ or $\frac{d}{dx}(k)=0.$

b. When $f(x)=x^2$, the quotient difference

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}$$
$$=\frac{(x^2+2xh+h^2)-x^2}{h}$$
$$=\frac{2xh+h^2}{h}=2x+h.$$

As h gets closer and closer to 0, the quantity $2x+h$ gets closer to $2x$. Therefore, by definition 2.6

$$f'(x)=2x \quad \text{or} \quad \frac{d}{dx}(x^2)=2x.$$

c. For $f(x)=\sqrt{x}$ the quotient

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}, \text{ using rationalization}$$
$$=\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$=\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

As h tends to 0, this quantity approaches $\frac{1}{2\sqrt{x}}$.

Hence, by definition 2.7,

$$f'(x)=(\sqrt{x})'=\frac{1}{2\sqrt{x}} \quad \text{or} \quad \frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}.$$

Exercise 2.7

Use definition 2.7, find the derivative of the function with respect to x.

a. $f(x)=x$ b. $f(x)=x^3$ c. $f(x)=\frac{1}{x}$ d. $f(x)=2\sqrt{x}$

2.1.4.3 Derivatives of various functions

i. Differentiation of power functions

Activity 2.7

Using definition 2.7, find the derivative of each of the following power functions.

1. a. $f(x)=x$ b. $f(x)=x^2$ c. $f(x)=x^{-2}$.
2. What is the derivative of $f(x)=x^n$?

Theorem 2.1

Let $f(x)=x^n$, where n is a positive integer. Then $f'(x)=nx^{n-1}$.

Proof

Let $f(x)=x^n$. The quotient difference

$$\frac{f(t)-f(x)}{t-x}=\frac{t^n-x^n}{t-x}$$

$$=\frac{(t-x)(t^{n-1}+xt^{n-2}+\ldots+tx^{n-2}+x^{n-1})}{t-x}$$

$$=t^{n-1}+xt^{n-2}+\ldots+tx^{n-2}+x^{n-1}, t\neq x$$

Thus, by definition of the derivatives as t gets very close to x

$$f'(x)=\underbrace{x^{n-1}+x^{n-1}+\ldots+x^{n-1}+x^{n-1}}_{\text{n-times}}=nx^{n-1}$$

Note: By long division, we have

$$t^n-x^n=(t-x)(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+x^{n-2}t+x^{n-1}).$$

i.e.,

$$
\begin{array}{l}
\qquad\quad t^{n-1}+xt^{n-2}+x^{2}t^{n-3}+\ldots+x^{n-2}t+x^{n-1} \\
t-x\,\overline{)\;t^{n} \qquad\qquad\qquad\qquad\qquad\quad -x^{n}} \\
\qquad\quad \underline{-(t^{n}-xt^{n-1})} \\
\qquad\qquad\quad xt^{n-1}-x^{n} \\
\qquad\qquad \underline{-(xt^{n-1}-x^{2}t^{n-2})} \\
\qquad\qquad\qquad x^{2}t^{n-2}-x^{n} \\
\qquad\qquad\quad \underline{-(x^{2}t^{n-2}-x^{3}t^{n-3})} \\
\qquad\qquad\qquad \underline{\quad\vdots} \\
\qquad\qquad\qquad x^{n-1}t-x^{n} \\
\qquad\qquad\quad \underline{-(x^{n-1}t-x^{n})} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

Example 1

Find the derivatives of each of the following functions.

a. $f(x)=x^{15}$ b. $f(x)=x^{24}$

Solution

a. $f'(x)=15x^{15-1}=15x^{14}$ b $f'(x)=24x^{24-1}=24x^{23}$

Corollary 2.1

If $f(x)=x^{-n}$, where n is a positive integer, then $f'(x)=-nx^{-(n+1)}$.

Example 2

Find the derivatives of each of the following functions.

a. $f(x)=x^{-5}$ b. $f(x)=x^{-17}$

Solution

a. $f'(x)=-5x^{-5-1}=-5x^{-6}$ b. $f'(x)=-17x^{-17-1}=-17x^{-18}$

Example 3

Let $f(x) = x^{-6}$, find

a. $f'(1)$ b. $f'(\frac{1}{3})$ c. $f'(k),\ k \neq 0.$

Solution

$f'(x) = -6x^{-7} \Rightarrow a)\ f'(1) = -6(1)^{-7} = -6$

$b)\ f'(\frac{1}{3}) = -6(\frac{1}{3})^{-7} = -6(3)^{7} = -13{,}122$

$c)\ f'(k) = -6(k)^{-7} = -6k^{-7}$

Corollary 2.2

Let $f(x) = kx^{n}$, then $f'(x) = knx^{n-1}$, where n is any non-zero integer and k is a constant number.

Example 4

Find the derivative of each of the following functions.

a. $f(x) = -2x^{11}$ b. $f(x) = 4x^{20}$ c. $f(x) = -\frac{3}{x^{7}}$ d. $f(x) = \frac{1}{3x^{15}}$

Solution

Using corollary 2.2, we obtain

a. $f'(x) = -2(11)x^{10} = -22x^{10}$ b. $f'(x) = 4(20)x^{19} = 80x^{19}$

c. $f'(x) = -3(-7)x^{-8} = 21x^{-8} = \frac{21}{x^{8}}$ d. $f'(x) = \frac{1}{3}(-15)x^{-16} = -\frac{5}{x^{16}}$

Exercise 2.8

1. Find the derivative of each of the following functions.

a. $f(x) = x^8$

b. $f(x) = x^{27}$

c. $f(x) = 2x^6$

d. $f(x) = 10x^{15}$

e. $f(x) = 4x^{-7}$

f. $f(x) = 3x^{-20}$

g. $f(x) = 4x^{-\frac{3}{2}}$

h. $f(x) = -10x^{-\frac{2}{5}}$

2. Let $f(x) = x^{-9}$, find

a. $f'(1)$

b. $f'\left(\frac{1}{3}\right)$

c. $f'(k) = k, k \neq 0$.

3. Based on Theorem 2.1, show the proof of corollary 2.1 and corollary 2.2.

ii. Derivatives of combinations of functions

a. Derivatives of a sum and difference of a functions

Activity 2.8

Given the functions $f(x) = x^2$, and $g(x) = x^3 + 2x$, then find

a. $f'(x) + g'(x)$

b. $(f + g)'(x)$

c. $f'(x) - g'(x)$

d. $(f - g)'(x)$

Theorem 2.2

If f and g are differentiable functions at a, then $f + g$ and $f - g$ are also differentiable at a and are given by $(f + g)'(a) = f'(a) + g'(a)$, and $(f - g)'(a) = f'(a) - g'(a)$

Proof

From the definition of derivative, we have:

$$\frac{(f+g)(x)-(f+g)(a)}{x-a}=\frac{f(x)+g(x)-f(a)-g(a)}{x-a}$$
$$=\frac{f(x)-f(a)+g(x)-g(a)}{x-a}$$
$$=\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a} \qquad (i)$$

By definition of derivatives, as x gets close to a,

$$\frac{(f+g)(x)-(f+g)(a)}{x-a}=(f+g)'(a),$$
$$\frac{f(x)-f(a)}{x-a}=f'(a) \text{ and } \frac{g(x)-g(a)}{x-a}=g'(a) \qquad \text{(ii)}$$

From Eq. (i) and Eq. (ii), we obtain

$$(f+g)'(a)=f'(a)+g'(a).$$

This theorem, implies for all x at which both f and g are differentiable,

$$(f+g)'(x)=f'(x)+g'(x).$$

Corollary 2.3

If $f_1, f_2, f_3, \ldots, f_n$ are differentiable at a number a, then

$$(f_1+f_2+f_3+\ldots+f_n)'(a)=f_1'(a)+f_2'(a)+f_3'(a)+\ldots+f_n'(a) \text{ and}$$

$$(f_1-f_2-f_3-\ldots-f_n)'(a)=f_1'(a)-f_2'(a)-f_3'(a)-\ldots-f_n'(a)$$

Example 1

If $f(x)=x^3+3x^2$, find a formula for $f'(x)$ and compute $f'(2)$.

Solution

$f'(x)=3x^2+6x \Rightarrow f'(2)=3\times 2^2+6\times 2=24.$

Example 2

Find the derivative of the function $f(x) = 3x^4 - 2x^3 + 10x + \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution

$$f'(x) = 12x^3 - 6x^2 + 10 + \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Exercise 2.9

1. If $f(x) = x^4 - 3x^2 + 2$, find a formula for $f'(x)$ and compute $f'(3)$.
2. Find the derivative of each of the following functions.

   a. $f(x) = 5x^4 + \sqrt{x}$ b. $f(x) = x + \frac{1}{x^2}$ c. $f(x) = \frac{1}{6}x^6 - 2x - 4$

   d. $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$ e. $f(x) = x^{10} + \frac{1}{x} + 6x^{\frac{5}{3}}$ f. $f(x) = 6x^{\frac{4}{3}} - \frac{1}{3x^3} - \frac{1}{2}x^2$
3. Based on the proof of the sum rule of derivative, show the proof of the difference rule.

b. Derivative of a constant multiple of a function

Activity 2.9

Given the function $f(x) = x^2$, find a. $(3f)'(x)$ b. $3(f'(x))$

Theorem 2.3

If f is differentiable at a, then for any constant number k, the function kf is differentiable at a and $(kf)'(a) = k(f'(a))$.

Proof

By definition of the derivative as $x \to a$,

$$(kf)'(a) = \frac{(kf)(x) - (kf)(a)}{x - a} = \frac{k[f(x) - f(a)]}{x - a} = k\frac{[f(x) - f(a)]}{x - a} = kf'(a).$$

Thus, for all x at which f is differentiable,

$$(kf)'(x) = kf'(x).$$

Example 1

Let $f(x)=6\sqrt{x}$, find the formula for $f'(x)$ and compute $f'(4)$.

Solution

$$f'(x)=(6\sqrt{x})'=\frac{6}{2\sqrt{x}}=\frac{3}{\sqrt{x}} \text{ and } f'(4)=\frac{3}{\sqrt{4}}=\frac{3}{2}.$$

Derivative of polynomial functions

If $f(x)=a_n x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$,, then

$$f'(x)=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+(n-2)a_{n-2}x^{n-3}+...+a_1.$$

Example 2

Given $y=3x^8-\sqrt{2}x^5+\frac{3}{2}x^3+20x+1$, compute $\left.\frac{dy}{dx}\right|_{x=-1}$.

Solution

Using differentiation rule

$$\frac{d}{dx}\left(3x^8-\sqrt{2}x^5+\frac{3}{2}x^3+20x+1\right)=24x^7-5\sqrt{2}x^4+\frac{9}{2}x^2+20$$

Then, we evaluate the derivative at $x=-1$.

$$\left.\frac{dy}{dx}\right|_{x=-1}=-24-5\sqrt{2}+\frac{9}{2}+20=\frac{1}{2}-5\sqrt{2}.$$

Note: $\left.\frac{dy}{dx}\right|_{x=-1}$ means value obtained by substituting $x=-1$ into $\frac{dy}{dx}$.

Exercise 2.10

1. Find the derivatives of the following functions.

a. $f(x)=\frac{1}{10}x^{10}$

b. $f(x)=3x^{\frac{4}{3}}$

c. $f(x)=\sqrt{3}x^3$

d. $f(x)=x^4-2x^3+3x-2$

2. Find the derivatives of the following functions, and compute the indicated values.

a. $y = 2\sqrt{x}$, evaluate $\left.\frac{dy}{dx}\right|_{x=1}$ and $\left.\frac{dy}{dx}\right|_{x=4}$.

b. $y = \frac{9}{x^3}$, evaluate $\left.\frac{dy}{dx}\right|_{x=-1}$ and $\left.\frac{dy}{dx}\right|_{x=3}$.

c. $y = x^3 + \frac{1}{2}x^2 - \frac{2}{\sqrt{x}}$, evaluate $\left.\frac{dy}{dx}\right|_{x=1}$ and $\left.\frac{dy}{dx}\right|_{x=4}$.

c. Derivative of a product

Activity 2.10

Given the functions $f(x) = x^2$, and $g(x) = x$, find

a. $(fg)'(x)$

b. $f'(x)g(x) + f(x)g'(x)$

Theorem 2.4

If f and g are differentiable at a, then fg is also differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof

By definition of the derivative as x gets very close to a,

$$\frac{(fg)(x) - (fg)(a)}{x - a} = (fg)'(a). \qquad (i)$$

and

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x - a} &= \frac{f(x)g(x) - f(a)g(a)}{x - a} \\ &= \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \\ &= \frac{g(x)\big(f(x) - f(a)\big) + f(a)\big(g(x) - g(a)\big)}{x - a} \\ &= \frac{g(x)\big(f(x) - f(a)\big)}{x - a} + \frac{f(a)\big(g(x) - g(a)\big)}{x - a} \\ &= g(x)\frac{\big(f(x) - f(a)\big)}{x - a} + f(a)\frac{\big(g(x) - g(a)\big)}{x - a}\end{aligned}$$

As $x \to a$, the expression $\frac{f(x)-f(a)}{x-a} = f'(a)$ and $\frac{g(x)-g(a)}{x-a} = g'(a)$. Thus,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

From this theorem, we generalize that, for all x at which both f and g are differentiable

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Example

1. Find the derivative of the following functions.

a. $g(x) = (x+6)(x^2+4)$ b. $f(x) = x^2(\sqrt{x}+2x)$

Solution

a. Using product property,

$$g'(x) = (x+6)'(x^2+4) + (x+6)(x^2+4)' = (x^2+4) + 2x(x+6) = 3x^2+12x+4.$$

b. Using product property,

$$f'(x) = \left(x^2\right)'(\sqrt{x}+2x) + x^2(\sqrt{x}+2x)'$$

$$= 2x(\sqrt{x}+2x) + x^2\left(\frac{1}{2\sqrt{x}}+2\right)$$

$$= 2x\sqrt{x} + 4x^2 + \frac{x^2}{2\sqrt{x}} + 2x^2$$

$$= 6x^2 + \frac{5x\sqrt{x}}{2}$$

Exercise 2.11

Find $f'(x)$ of the following functions using product rule.

a. $f(x) = (2x+1)(x^2+x-5)$

b. $f(x) = (x^2-2)(x^3+4x)$

c. $f(x) = x(x^2+\sqrt{x})$

d. $f(x) = (x+2\sqrt{x})(x^2-2)$

d. Derivative of a quotient

Theorem 2.5

If f and g are differentiable at a, and $g(a) \neq 0$, then $\frac{1}{g}, \frac{f}{g}$ are differentiable at a and

i. $$\left(\frac{1}{g}\right)'(a) = \frac{(1)'g(a) - 1(g'(a))}{(g(a))^2} = -\frac{g'(a)}{(g(a))^2}$$

ii. $$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof

i. By definition of derivative as x gets very close to a,

$$\frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \left(\frac{1}{g}\right)'(a)$$

And $\left(\frac{1}{g}\right)'(a) = \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \frac{g(a)-g(x)}{(x-a)g(x)g(a)} = -\frac{g(x)-g(a)}{x-a} \cdot \frac{1}{g(x)g(a)}$

As $x \to a$, the quantity $\frac{g(x)-g(a)}{x-a} = g'(a)$

Thus $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}$

ii. $\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a)$

Using product rule $\left(f \cdot \frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$

$$= f'(a)\frac{1}{g(a)} + f(a)\left(-\frac{g'(a)}{(g(a))^2}\right)$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$$

From this theorem, we see that, for all x at which both f and g are differentiable

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Example

Find $f'(x)$ a. $f(x)=\dfrac{1}{x^2+3x-1}$ b. $f(x)=\dfrac{8x^5}{x^3+2x}$

Solution

Using quotient rule

a. $f'(x)=-\dfrac{\left(x^2+3x-1\right)'}{\left(x^2+3x-1\right)^2}=-\dfrac{2x+3}{\left(x^2+3x-1\right)^2}$

b. $f'(x)=\dfrac{(8x^5)'(x^3+2x)-8x^5(x^3+2x)'}{(x^3+2x)^2}$

$=\dfrac{(40x^4)(x^3+2x)-8x^5(3x^2+2)}{(x^3+2x)^2}=\dfrac{40x^7+80x^5-24x^7-16x^5)}{(x^3+2x)^2}$

$=\dfrac{16x^7+64x^5}{(x^3+2x)^2}.$

Exercise 2.12

Find the derivative of each of the following functions.

a. $f(x)=\frac{1}{x^2+1}$

b. $f(x)=\dfrac{2x^3+3}{x-5}$

c. $f(x)=\frac{x^3+3x}{x^2}$

d. $f(x)=\frac{2x^4}{\sqrt{x}}$

2.1.4.4 The chain rule

Activity 2.11

1. Let $f(x)=2x+3$ and $g(x)=x^2-1$. Find each of the following functions.

   a. $f'(x)$ b. $f'(g(x))$ c. $f(g(x))$ d. $f'(g(x))\cdot g'(x)$

2. Find the derivatives of the following functions.

   a. $f(x)=\left(3x^2+2\right)^2$ b. $f(x)=\left(\sqrt{x^3+2x-7}\right)^3$

Theorem 2.6

Let g be differentiable at a, and f be differentiable at $g(a)$ .Then $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

The Chain rule can be interpreted as saying that $(f \circ g)'(x) = f'(g(x))g'(x)$ for all x such that g is differentiable at x and f is differentiable at $g(x)$.

Proof

For $g(x) - g(a) \neq 0$,

$$\frac{f(g(x)) - f(g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)}$$

By definition of derivative as x gets very close to a,

$$\frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a}$$

$$= \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

As $x \to a$, the expressions

$$\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)) \text{ and } \frac{g(x) - g(a)}{x - a} = g'(a)$$

Thus, $(f \circ g)'(a) = f'(g(a))\, g'(a)$.

Example 1

For each of the following functions find a formula for $f'(x)$.

a. $f(x) = \sqrt{x^4 + 3x^2 - 5}$ b. $f(x) = (5x + 4)^{21}$

Solution

Using the Chain rule

a. f is the composition of two simple functions $g(x) = \sqrt{x}$ and $h(x) = x^4 + 3x^2 - 5$. That is, $f(x) = g(h(x))$.

By the chain rule, $f'(x) = g'(h(x))\,h'(x)$.

But $g'(x) = \dfrac{1}{2\sqrt{x}}$ and $h'(x) = 4x^3 + 6x$

Thus, $f'(x) = g'(h(x))\,h'(x) = \dfrac{1}{2\sqrt{x^4+3x^2-5}} \cdot \left(4x^3+6x\right) = \dfrac{2x^3+3x}{\sqrt{x^4+3x^2-5}}$.

b. $f'(x) = 21(5x+4)^{21-1} = 21(5x+4)^{20}(5) = 105(5x+4)^{20}$.

Example 2

Let $y = \sqrt{4+x^5}$. Find a formula for $\dfrac{dy}{dx}$ and compute $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Solution

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4+x^5}}\frac{d}{dx}(x^5+4) = \frac{5x^4}{2\sqrt{4+x^5}} \text{ and } \left.\frac{dy}{dx}\right|_{x=2} = \frac{20}{3}.$$

Exercise 2.13

Find the derivative of the following functions using the chain rule.

a. $f(x) = (2x+1)^4$

b. $f(x) = (x^3 + 2x)^{25}$

c. $f(x) = \sqrt{x^2+3x-1}$

d. $f(x) = \frac{1}{(3x+2)^2}$

e. $f(x) = \sqrt{x^2 + \dfrac{1}{x^2}}$

f. $f(x) = \left(\frac{3x-5}{2-7x}\right)^6$

2.1.5 Maximum and Minimum Points

The differentiation of various functions renders great services in solving the problems concerned with finding out the maximum and minimum values of quantities. In the various fields of engineering and technology, we have to find the maximum or minimum values of one quantity with respect to another quantity. For example, in finding the radius and height of a cylinder that is to be manufactured with the metal sheet of a given surface area, so that the capacity of cylinder is maximum. Sometimes it is necessary to find the least cost for the transmission of given horsepower. We can find the dimensions of a plot of a given perimeter so that its area is maximum and so on.

2.1.5.1 Increasing and Decreasing Functions

Activity 2.12

1. Find the real zeros of each of the following functions.

   a. $f(x) = 4x - 8$ b. $f(x) = x^2 - x - 12$ c. $f(x) = \frac{1-\sqrt{x}}{(x-2)^2}$

2. Consider the graph of the following function.

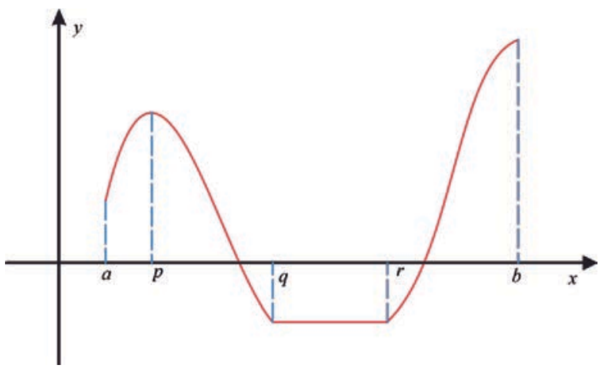


Figure 2.10

   a. Identify the intervals in which the graph is rising and falling as you move from left to right starting at *a*.
   b. Identify the intervals in which the graph is neither rising nor falling.

3. Solve each of the following inequalities using sign chart method:

   a. $2x^2 + 3x - 2 \geq 0$ b. $x^2 + 3x - 4 < 0$

Definition 2.8

Let f be a function on an interval I. If x_1 and x_2 are in I,

i. For $x_1 < x_2$ if $f(x_1) \le f(x_2)$, then f is said to be increasing on I.

ii. For $x_1 < x_2$ if $f(x_1) \ge f(x_2)$, then f is said to be decreasing on I.

iii. For $x_1 < x_2$ if $f(x_1) < f(x_2)$, then f is strictly increasing on I.

iv. For $x_1 < x_2$ if $f(x_1) > f(x_2)$, then f is strictly decreasing on I.

Geometrically, a function is increasing if its graph rises and decreases if its graph falls while x moves to the right.

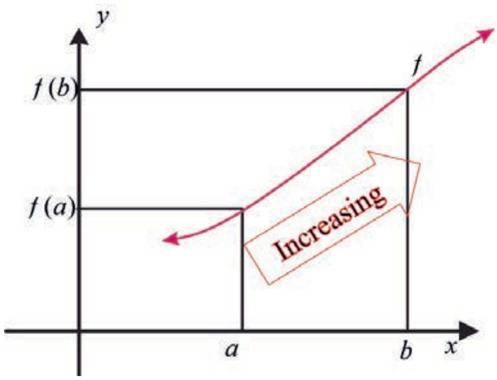


Figure 2.11a

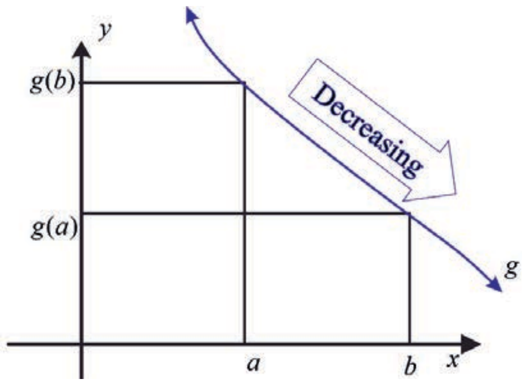


Figure 2.11b

Increasing and decreasing test

Suppose that $f(x_1) \ge f(x_2)$, f is differentiable in the interior of an interval I.

a. If $f'(x) \ge 0$, for all x in the interior of I, then f is increasing on I.

b. If $f'(x) \le 0$, for all x in the interior of I, then f is decreasing on I.

c. If $f'(x) > 0$ and $f'(x) = 0$ on I, then f is strictly increasing on I.

d. If $f'(x) < 0$ and $f'(x) = 0$ on I, then f is strictly decreasing on I.

Note

A function that is either increasing or decreasing is known as a monotonic function.

Definition 2.9

A number c in the domain of a function f is said to be a **critical number** of f if and only if either $f'(c)=0$ or f has no derivative at c.

Example 1

Find the critical numbers of the given functions.

a. $f(x)=4x^3-5x^2-8x+20$

b. $f(x)=2\sqrt{x}(6-x)$

Solution

a. $f'(x)=12x^2-10x-8$

$$f'(x)=0 \Rightarrow 2(6x^2-5x-4)=2(2x+1)(3x-4)=0 \Rightarrow x=-\frac{1}{2} \text{ or } x=\frac{4}{3}$$

Hence, the critical numbers of f are $x=-\frac{1}{2}$ and $x=\frac{4}{3}$.

b. Applying the product rule of derivatives:

$$f'(x)=\frac{2}{2\sqrt{x}}(6-x)-2\sqrt{x}=\frac{6-x}{\sqrt{x}}-\frac{2x}{\sqrt{x}}=\frac{6-3x}{\sqrt{x}}=\frac{3(2-x)}{\sqrt{x}} \Rightarrow$$

$f'(x)=0 \Rightarrow x=2$

and $f'(x)$ does not exist when $x=0$. Hence, the critical numbers of f are $x=0$ and $x=2$.

Example 2

Find the interval on which the function $f(x)=3x^4-4x^3-12x^2+5$ is increasing and decreasing.

Solution

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$

Hence, the critical numbers f are $x = -1, 0, 2$

		-1		0		2	
$12x$	----		----	0	++++		+++++
$x+1$	----	0	++++		++++		++++
$x-2$	----		-----		------	0	++++
$f'(x) = 12x(x-2)(x+1)$	----	0	++++	0	-----	0	++++

From the sign chart above,

$f'(x) \geq 0$ on $[-1,0] \cup [2,\infty)$ and $f'(x) \leq 0$ *on* $(-\infty,-1] \cup [0,2]$.

Therefore, the function is increasing on an interval $[-1,0] \cup [2,\infty)$ and decreasing on $(-\infty,-1] \cup [0,2]$.

Exercise 2.14

1. Find all the critical numbers of the following functions.

   a. $f(x) = x^3 + 6x^2$

   b. $f(x) = \dfrac{2x^2}{1-x^2}$

   c. $f(x) = 2x^3 + 9x^2 - 24x$

   d. $f(x) = 8x\sqrt{1-x^2}$

2. Find the interval on which the following functions are increasing and decreasing.

   a. $f(x) = x^3 + 6x^2$

   b. $f(x) = 2x^2 + 6x - 10$

   c. $f(x) = -2x^3 + 3x^2 + 12x + 6$

   d. $f(x) = x - 6\sqrt{x-1}$

   e. $f(x) = (x+3)^2(x-1)^2$

2.1.5.2 Minimum and maximum values of a function

Activity 2.13

Consider the graph of the following function.

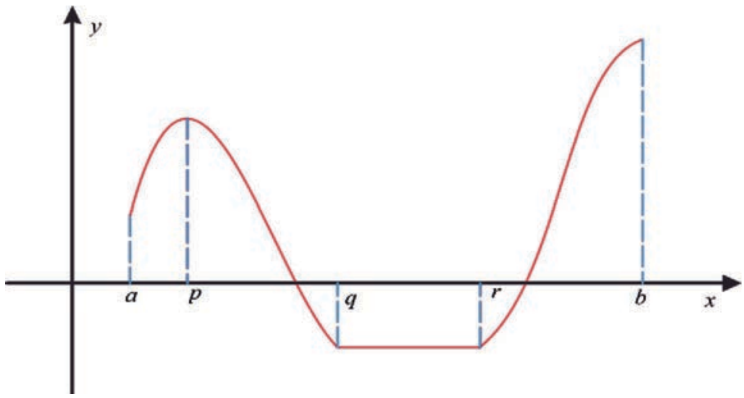


Figure 2.12

a. Where does the function attain maximum value and a minimum value?

b. What happens to the gradient at a maximum point and at a minimum point?

Definition 2.10

If there exists an open interval (a,b) containing c such that $f(x) < f(c)$ for all x in (a,b) other than c in the interval, then $f(c)$ is a relative (local) maximum value of f and a point $(c, f(c))$ is called relative maximum point. If $f(x) > f(c)$ for all x in (a,b) other than c, then $f(c)$ is a relative (local) minimum value of f and the point $(c, f(c))$ is known as relative(local) minimum point (See Figure 2.13 below). Functions may have any number of relative extrema.

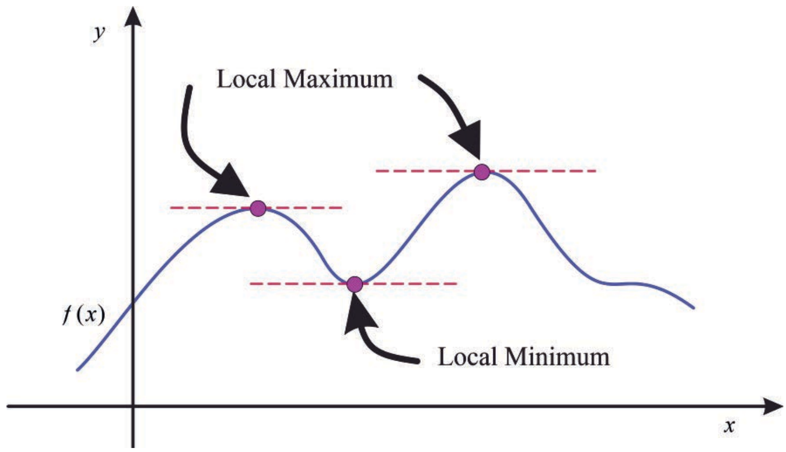


Figure 2.13

Definition 2.11

If c is in the domain of f and for all x in the domain of the function, $f(x) \leq f(c)$, then $(c, f(c))$ is an absolute maximum piont of the function f. If for all x in the domain $f(x) \geq f(c)$ then $(c, f(c))$ is an absolute minimum point of the function f. (see Figure 2.14). The absolute maximum and minimum values of f are called absolute extreme values.

Note

- Absolute extrema are not necessarily unique.
- At either local maximum or local minimum point the gradient is zero.

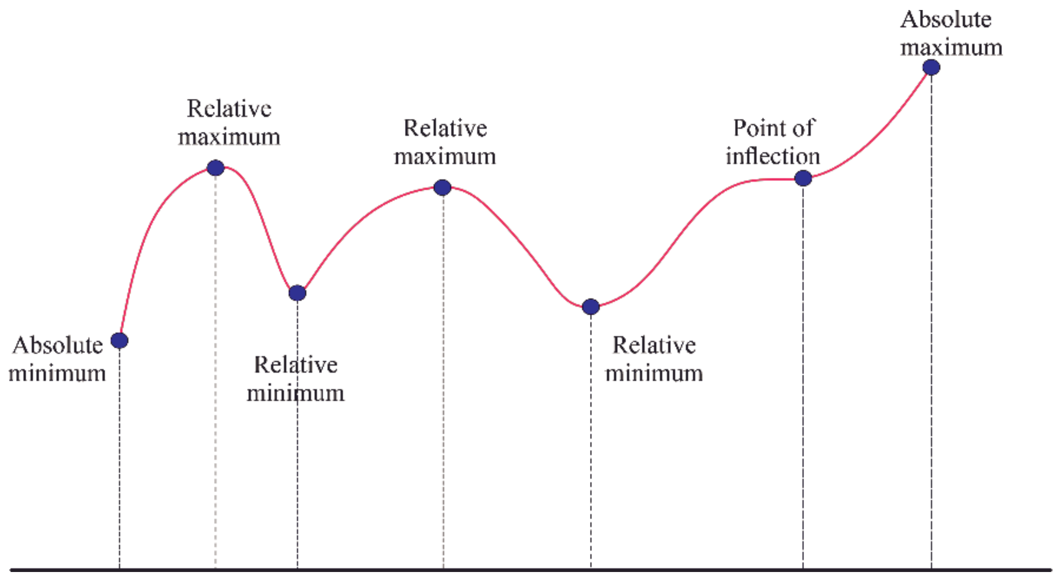


Figure 2.14

Procedure to find absolute extreme values on closed intervals

We can find the absolute maximum and minimum values of a function f on a closed interval $[a,b]$ by the following steps:

Step 1: Find all the critical numbers of a function f.

Step 2: Evaluate f at all critical numbers and at the end points a and b.

Step 3: Choose the largest and the smallest values obtained in step2. These values are the maximum and the minimum of a function f on $[a,b]$.

Example

Find the absolute maximum and minimum values of the function $f(x) = x^2 - 2x$ on $[-1,2]$.

Solution

$f'(x) = 2x - 2 = 2(x-1)$

$$f'(x) = 0 \Rightarrow x = 1.$$

Thus, the critical number of f is $x = 1$.

Thus, to determine the extreme value of the function we need to evaluate the value of the function at the boundary and critical number.

$$f(-1)=1+2=3\ ,\ f(1)=1-2=-1 \text{ and } f(2)=4-4=0.$$

Therefore, the absolute maximum value is $f(-1)=3$ and the absolute minimum value is $f(1)=-1$. Thus, $(-1,3)$ is the maximum point and $(1,-1)$ is the minimum point of f.

Exercise 2.15

Find the absolute maximum and minimum values of the following functions on a given interval.

a. $f(x)=x^2+2x+3,\quad [-2,2]$

b. $f(x)=x^3+3x^2-9x+5,[-2,2]$

c. $f(x)=3x^4-26x^3-60x^2-11$ on $[1,5]$.

d. $f(x)=\dfrac{x^2-2x+4}{x-2},[-3,1]$

The first derivative test

Let c be a critical number of a function f on an open interval I containing c.

a. If the sign of f'changes from negative to positive at c, then f has a relative minimum at c.

b. If the sign of f'changes from positive to negative at c, then f has a relative maximum at c.

c. If the sign of f' does not change at c, then f has no relative maximum or minimum at c.

Procedure to find relative extreme value

To determine relative extreme value of a function using the first derivative test, one can follow the following steps.

Step 1: Find all the critical numbers of f.

Step 2: Form the sign chart for $f'(x)$ using the critical number as poles.

Step 3: Apply first derivative test and decide the extreme values if any.

Example 1

Find the relative maximum and minimum values of

a. $f(x)=2x^3+3x^2-12x-3$ b. $f(x)=(x-1)^2(x-3)^2$

Solution

a. $f'(x)=6x^2+6x-12=6(x^2+x-2)=6(x+2)(x-1)$

Hence, the critical numbers are $x=-2,1$

		-2		1	
$6(x+2)$	----	0	++++		+++++
$x-1$	----		----	0	+++++
$f'(x)=6(x+2)(x-1)$	+++++	0	----	0	+++++

From the sign chart above we see that, f' changes its sign from positive to negative at $x=-2$ and changes from negative to positive $x=1$, therefore, the function has a local maximum value at $x=-2$ and a local minimum value at $x=1$. This implies the point $(-2, f(-2))=(-2,17)$ is a relative maximum and $(1, f(1))=(1,-10)$ is a relative minimum point of the function.

Thus, 17 is a relative maximum value and – 10 is a relative minimum value.

b.

$$\begin{aligned} f'(x) &= 2(x-1)(x-3)^2 + 2(x-1)^2(x-3) \\ &= 2(x-1)(x-3)(x-3+x-1) \\ &= 2(x-1)(x-3)(2x-4) \\ &= 4(x-1)(x-2)(x-3) \end{aligned}$$

Hence, the critical numbers are $x = 1, 2, 3$.

		1		2		3	
$4(x-1)$	----	0	++++		++++		+++++
$x-2$	----		-----	0	++++		++++
$x-3$	----		-----		------	0	++++
$f'(x) = 4(x-1)(x-2)(x-3)$	----	0	++++	0	-----	0	++++

From the sign chart, we see that f' changes its sign from negative to positive at $x = 1$ and $x = 3$ and changes from positive to negative at $x = 2$. Therefore, $(1, f(1)) = (1, 0)$ and $(3, f(3)) = (3, 0)$ are relative minimum points and $(2, f(2)) = (2, 1)$ is the relative maximum point of the function.

Thus, 0 is a relative maximum value and 1 is a relative minimum value.

Example 2

If $f(x) = ax^3 + bx - 5$ has a relative minimum value of -6 at $x = \frac{1}{2}$, find a and b.

Solution

Given $f(\frac{1}{2}) = \frac{1}{8}a + \frac{1}{2}b - 5 = -6 \Leftrightarrow a + 4b = -8$ (i)

and $f'(\frac{1}{2}) = 0$, $f'(x) = 3ax^2 + b \Rightarrow \frac{3a}{4} + b = 0 \Leftrightarrow 3a + 4b = 0$ (ii)

(i) and (ii) imply $\begin{cases} a + 4b = -8 \\ 3a + 4b = 0 \end{cases} \Rightarrow a = 4 \;\; and \; b = -3.$

Exercise 2.16

1. Find the relative extreme value of the following functions (if any).

   a. $f(x)=2x^3-9x^2-24$ b. $f(x)=\frac{1}{4}x^3-3x$ c. $f(x)=\frac{x^2+x+1}{x^2-x+1}$

2. If $f(x)=x^3+ax+b$ has a relative minimum value of 4 at $x=2$, find a and b.

2.1.6 Equation of Tangents and Normal Lines to Curves

Activity 2.14

1. Find the equation of a line containing the points $(1,3)$ and $(-2,3)$.
2. Find the equation of a line perpendicular to the line found in question 1 above containing the point $(1,3)$.
3. Write the equation of a tangent line to a curve if its slope is 3 at the point $(-2,5)$.
4. What is the relationship between the slopes of a tangent line and a normal line to a curve at a point of tangency?

Is there any relation ship between the first derivative of a function at a point of tengency and the slope of a tangent line to the curve of the function?

We have seen in section 2.1.5 that the first derivative of a function at a given point is interpreted as the slope of the tangent line at that point. This implies f' is the slope of the line tangent to the graph of f at a point $(a, f(a))$. Thus, the equation of the tangent line to the curve $y=f(x)$ at $x=a$ is given by $y=f'(a)(x-a)+f(a)$.

A line that is perpendicular to the tangent line at the point of tangency is known as the **normal line** to the curve. Since the product of the slope of two non-vertical perpendicular lines is -1, the slope of the normal line at the point of tangency $(a, f(a))$ is $\frac{-1}{f'(a)}$ and its equation is given by $y=-\frac{1}{f'(a)}(x-a)+f(a)$.

Note

1. If the first derivative of f at $x = a$ is zero, then f has a horizontal tangent line with equation $y = f(a)$ and vertical normal line given by $x = a$.
2. If the first derivative of f at $x = a$ undefined, then f has a vertical tangent line with equation $x = a$ and a horizontal normal line given by $y = f(a)$.

Example 1

Find the equation of the line tangent and the normal line to the graph of the following functions at the given point.

a. $f(x) = x^3 - 2x;\ (1,-1)$

b. $f(x) = \sqrt{20 - x^2}; (2,4)$

c. $f(x) = \begin{cases} x, & \text{if } x > 3 \\ x^2\text{-}6, & \text{if } x \le 3 \end{cases}$

d. $f(x) = x\sqrt{x^2 + 16}\ ; (0,0)$

Solution

a. $f'(x) = 3x^2 - 2 \Rightarrow f'(1) = 1$

Thus,

i. The equation of the tangent line to the graph of f at (1,-1) is

$y = f'(1)(x-1) + f(1) = 1(x-1) - 1 = x - 2. \Leftrightarrow y = x - 2.$

ii. The equation of the normal line to the graph of f at (1,-1) is

$$
\begin{aligned}
y &= -\frac{1}{f'(1)}(x-1) + f(1) \\
&= -1(x-1) - 1 \\
&= -x.
\end{aligned}
$$

$\Leftrightarrow y = -x$

b. $f'(x) = \dfrac{-x}{\sqrt{20 - x^2}} \Rightarrow f'(2) = \dfrac{-1}{2}$

i. The equation of the tangent line to the graph of f at (2,4) is $y = f'(2)(x-2) + f(2) = -\frac{1}{2}(x-2) + 4 = -\frac{1}{2}x + 5$

$\Leftrightarrow y = -\frac{1}{2}x + 5.$

ii. Since the slope of a normal line is negative reciprocal of that of the tangent line, the slope of the normal line in this problem is 2. Hence, the equation of the normal line to the graph of f at (2,4) is

$$y = \frac{-1}{f'(2)}(x-2) + f(2) = 2(x-2) + 4 = 2x.$$

$\Leftrightarrow y = 2x.$

c. $f'(x) = \begin{cases} 1, \text{ if } x > 3 \\ 2x, \text{ if } x > 3 \end{cases} \Rightarrow f'(1) = 2.$

i. The equation of the tangent line to the graph of f at (1,-5) is

$y = f'(1)(x-1) + f(2) = 2(x-1) - 5 = 2x - 7.$

$\Leftrightarrow y = 2x - 7.$

ii. The equation of the normal line to the graph of f at (1,-5) is

$$y = -\frac{1}{f'(1)}(x-1) + f(1) = \frac{-1}{2}(x-1) - 5 = \frac{-1}{2}x - \frac{9}{2}.$$

$\Leftrightarrow y = -\frac{1}{2}x - \frac{9}{2}.$

d. $f'(x) = \sqrt{x^2+16} + \frac{x^2}{\sqrt{x^2+16}} \Rightarrow f'(0) = 4$

i. The equation of the tangent line to the graph of f at $(0,0)$ is

$y = f'(0)(x-0) + f(0) = 4(x-0) + 0 = 4x.$

$\Leftrightarrow y = 4x.$

ii. The equation of the normal line to the graph of f at $(0,0)$ is

$$y = \frac{-1}{f'(0)}(x-0) + f(0) = \frac{-1}{4}(x-0) = \frac{-1}{4}x.$$

$$\Leftrightarrow y = -\frac{1}{4}x.$$

Example 2

Find all points at where $f(x)=\sqrt{x^2-6x}$ has vertical tangent line.

Solution

The function will have a vertical tangent line at its first derivative is undefined in its domain and $f'(x)=\dfrac{2x-6}{2\sqrt{x^2-6x}}=\dfrac{x-3}{\sqrt{x^2-6x}}$ is undefined at the points where $x(x-6)=0 \Rightarrow x=0$ or $x=6$. Hence, $f(x)$ has a vertical tangent line at $(6,0)$ and $(0,0)$.

Exercise 2.17

1. Find the equation of the line tangent and the normal line to the graph of the following functions at the given point.

   a. $f(x)=x^3+x+2;\ (0,2)$ b. $f(x)=(1-x^3)\sqrt{x+2};\ (-1,2)$.

2. Find all points at where the following functions has vertical tangent line.

   a. $f(x)=\sqrt{x^2-x}$ b. $f(x)=2\sqrt{x}$

3. If $f(x)=x^2+ax+b$ and $g(x)=x^3-c$ have the same tangent line at $(1,2)$ find the values of the constants a,b and c and the equation of their normal lines.

2.2 Application of Derivative

Activity 2.15

1. Given a set $S = \{3,4,5,6,7\}$ and $f(x) = 3x + 2$

   a. Find $K = \{f(x) \mid x \in S\}$.

   b. What is the smallest element of K?

   c. What is the largest element of K?

2. Given a set S on an open interval $(3,7)$ and $f(x) = 3x + 2$

   a. Find $T = \{f(x) \mid x \in S\}$.

   b. Can you list all the elements of T?

   c. Can you guess the smallest element of T?

   d. Can you guess the largest element of T?

3. Given a set S as a closed interval $[3,7]$ and $f(x) = 3x + 2$

   a. Find $H = \{f(x) \mid x \in S\}$.

   b. Can you list all the elements of H?

   c. Can you guess the smallest element of H?

   d. Can you guess the largest element of H?

Applications of derivatives are varied not only in mathematics but also in real life. To give an example, derivatives have various important applications in mathematics such as finding the rate of change of a quantity, finding the approximate value, finding the equation of the tangent line and the normal line to a curve, and finding the minimum and maximum values of algebraic expressions. Derivatives also has a wide range of applications in real-life usage. For instance, in business, derivatives are used to find profit and loss for the future of the investment using graphs. It is used to calculate the rate of change of distance of a moving body with respect to time. Derivatives can have different interpretations in each of the sciences. For instance; chemists who study a chemical reaction may be interested in the rate of change in the concentration of a reactant with respect to time (the rate of reaction). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In section 2.1, you have studied derivatives and have developed methods to find derivatives of various functions. Once you have developed the properties of the mathematical concepts once and for all, you can then turn around and apply these results to all of the sciences. You have already investigated some of the applications of derivatives, but now that you know the differentiation rules, you are in a better position to pursue the applications of differentiation in a greater depth.

In this section, you will study problems involving variables that are changing with respect to time. If two or more such variables are related to each other, then their rates of change with respect to time are also related.

2.2.1 Applications of Derivatives in Finding Rate of Change

Example 1

If the radius of a circle is increasing at a rate of 1.5 cm/sec., find the rate at which the area is increasing at the instant when the diameter is 12 cm.

Solution

We know that area of a circle with radius r is given by $A = \pi r^2$ and we are given $\frac{dr}{dt} = 1.5\text{cm/sec}$, to find $\frac{dA}{dt}$ at diameter $d = 12\text{cm}$.

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \pi \frac{d(r^2)}{dt} = \pi(2r\frac{dr}{dt}) = 2\pi r\frac{dr}{dt}$$

Besides when the diameter $d = 12\text{cm}$, $r = 6\text{cm}$.

Hence, $\frac{dA}{dt}(6) = 2\pi(6)(1.5) = 18\pi\text{cm}^2/\text{sec}$.

Example 2

An object is moving along the path $3y = x^2$. If the x-coordinate of the object is increasing at a rate of 2 ft/min at the instant when $x = 6$, find the rate at which the y-coordinate is increasing.

Solution

Given $3y = x^2$ and $\frac{dx}{dt} = 2\text{ft/min}$

$$3y = x^2 \Rightarrow 3\frac{dx}{dt} = 2x\frac{dx}{dt} \Leftrightarrow \frac{dy}{dt} = \frac{2x}{3}\frac{dx}{dt}$$

Thus, at $x = 6$, $\frac{dy}{dt} = \frac{2}{3}(6)(2 \text{ ft/min})=8 \text{ ft/min}$

Example 3

Suppose the side of an equilateral triangle is increasing at a rate of 2 cm/sec. Find the rate at which the area is increasing at the instant when the side is 4 cm.

Solution

The area of an equilateral triangle having one side of lenth s is given by $A = \frac{\sqrt{3}}{4}s^2$

We are given, $\frac{ds}{dt} = 2\text{cm/sec}.$

To find $\frac{dA}{dt}$ when $s = 4\text{cm}$

$$A = \frac{\sqrt{3}}{4}s^2 \Rightarrow \frac{dA}{dt} = 2s\frac{\sqrt{3}}{4}\frac{ds}{dt}$$

$$\left.\frac{dA}{dt}\right|_{s=4} = 2\times\frac{\sqrt{3}}{4} = 4\sqrt{3}\ \text{cm}^2/\text{sec}$$

Example 4

Air is being pumped into a spherical balloon at the rate of $4\ \text{cm}^3/\text{min}$. Find the rate of change of the radius when the radius is 2 cm.

Solution

Let r be the radius of the sphere. The volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi} cm/\min.$$

Example 5

The displacement of a particle at any time t is given by $S = t^3 - 3t^2 + 5t + 7$. Find its velocity and acceleration at the end of 2 hrs, where S is in kilometers and t is in hrs.

Solution

Given $S = t^3 - 3t^2 + 5t + 7$

Velocity $v = \frac{dS}{dt} = 3t^2 - 6t + 5$

Acceleration $a = \frac{dv}{dt} = 6t - 6$

Hence, the velocity at the end of 2 hrs is

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2\text{hrs}} = 3(2)^2 - 6(2) + 5 = 5 \text{ km/hr}.$$

Acceleration at the end of 2 hrs is $a = \frac{dv}{dt} = 6(2) - 6 = 6$ km/hr^2.

Exercise 2.18

1. A circular plate of metal is being expanded by heating. The radius of plate increases at the rate of 0.3 cm per second. Find the rate of increase of area when its radius becomes 15 cm.

2. The volume of a sphere is increasing at the rate of $0.2\pi\ \text{cm}^3/\text{sec}$. Find the rate of increase of radius when the radius is 10 cm.
3. A square plate is being expanded by heating. If its side is increasing at the rate of 0.2 cm per second, then at what rate the area is increasing when the side is 20 cm?
4. A man 2 m tall walks at a uniform speed of 8 m/minute from a lamp post 6 m high. Find the rate at which the length of his shadow increases.
5. A varying force is applied on a body of unit mass and if the force increases at the rate of 2 N/sec., then find the rate at which the acceleration in the body is changing.

2.2.2 Applications of Derivatives in Business and Economics

Activity 2.16

1. Where do you think we use the concept of derivative other than in mathematics?
2. Is the concept of derivative vital in the study of economics, road taxation and traffic education?

From this activity, one can see the development of multiple strategies, into which companies incorporate derivatives. Companies, both in and out of the financial industry have begun to use derivatives as a method of speculating and generating income.

Example 1

The profit function of a company can be represented by $P(x) = x - 0.00001x^2$, where x represents units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

Solution

Marginal profit: $MP = \frac{dP}{dx} = \frac{d}{dx}(x - 0.00001x^2) = 1 - 0.00002x.$

To get maximum profit now we put marginal profit equals to zero. So,

$1 - 0.00002x = 0 \Rightarrow x = \frac{1}{0.00002} = 50,000$ units.

Now, to check whether this value is maximum or minimum we can use the first derivative test.

		50,000	
$P'(x) = 1 - 0.00002x$	++++	0	----

From this sign chart the sign of $P'(x)$ changes from positive to negative at $x = 50,000$. Thus, by the first derivative test, we get maximum profit when the company sold $x = 50,000$ units of production. Now, by putting the value of x in the profit function we get maximum profit.

$$P = f(50,000) = 50,000 - 0.00001(50000)^2$$
$$= 50,000 - 0.00001(2500,000,000) = 50,000 - 25,000 = 25,000$$

The optimum output for the company will be 50,000 units of x and the maximum profit at that volume will be 25, 000 birr.

Example 2

The demand equation for a certain product is $P = 6 - \frac{1}{2}x$ birr, where x represents the amount of product. Find the level of production which results in maximum revenue.

Solution

The revenue functionis $R(x)$.

Thus, $R(x) = x\left(6 - \frac{1}{2}x\right) = 6x - \frac{1}{2}x^2$ birr.

The marginal revenue is given by $R'(x) = 6 - x$

The graph of $R(x)$ is a parabola that opens downward as shown in Figure 2.15.

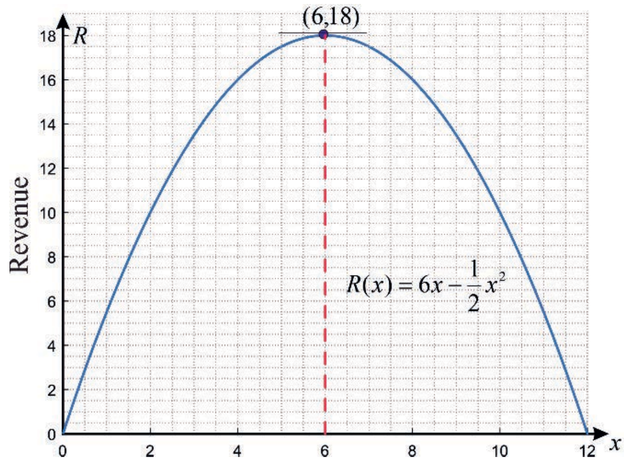


Figure 2.15

It has a horizontal tangent precisely at that x for which $R'(x) = 0$, that is, for that x at which marginal revenue is 0. The only such value of x is $x = 6$. The corresponding value of revenue is $R(6) = 6(6) - \frac{1}{2}(6)^2 = 18$ birr.

		6	
$R'(x) = 6 - x$	++++	0	----

Thus, by the first derivative test, the rate of production resulting in maximum revenue is $x = 6$, which results in total revenue of 18 birr.

Example 3

Promoters of international fund-raising concerts must walk a fine line between profit and loss, especially when determining the price to charge for admission to closed-circuit TV showings in local theatres. By keeping records, a theatre determines that at an admission price of 26 birrs, it averages 1000 people in attendance. For every drop in price of 1 birr, it gains 50 customers. Each customer spends an average of 4 birrs on concessions. What admission price should the theatre charge to maximize total revenue?

Solution

Let x be the number of birr by which the price of 26 Birr should be decreased. (If x is negative, the price is increased.) We first express the total revenue R as a function of x. Note that the increase in ticket sales is $50x$ when the price drops x Birr:

$$\begin{aligned} R(x) &= \text{(revenue from tickets)+(revenue from concessions)} \\ &= (number\ of\ \ people)\cdot(Ticket\ \ price)+(number\ of\ \ people)\cdot 4 \\ &= (1000+50x)\cdot(26-x)+(1000+50x)\cdot 4 \\ &= 26,000-1000x+1300x-50x^2+4000+200x \\ &= -50x^2+500x+30,000 \end{aligned}$$

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$$R'(x) = -100x+500$$

This derivative exists for all real numbers x. Thus, the only critical values are where $R'(x)=0$; so we solve that equation:

$$-100x+500=0 \Rightarrow x=5$$

This corresponds to lowering the price by 5 Birr. To determine whether *R(x)* has a maximum value or not at this critical point we use the first derivative test.

		5	
$R'(x) = 500 - 100x$	++++	0	----

Since the sign chart changes from negative to positive at $x=5$, by first derivative test, in order to maximize revenue, the theatre should charge 26 Birr-5 Birr or 21 Birr per ticket.

Example 4

For a company, the total revenue R and the total cost C for yard maintenance of x homes are given by $R(x)=1000x-x^2$, and $C(x)=3000+20x$.

Suppose that the company is adding 10 homes per day at the moment when the 400th customer is signed. At that moment, what is the rate of change of

a. total revenue? b. total cost? c. total profit?

Solution

a. $\frac{dR}{dt} = \frac{dR}{dx}\frac{dx}{dt} = (1000 - 2x)\frac{dx}{dt}$

$\left.\frac{dR}{dt}\right|_{x=400} = (1000 - 2\times 400)(10) = 2000$ birr per day.

b. $\frac{dC}{dt} = \frac{dC}{dx}\frac{dx}{dt} = (20)\frac{dx}{dt} = 20\times 10 = 200$ birr per day.

c. Since $p(x) = R(x) - C(x)$, we have

$\frac{dp}{dt} = \frac{dR}{dt} - \frac{dC}{dt} = 2000$ birr per day - 200 birr per day

$= 1800$ birr per day.

Exercise 2.19

Given $C(x) = 62x^2 + 27,500$ and $R(x) = x^3 - 12x^2 + 40x + 10$ where $C(x)$ and $R(x)$ represent, respectively, the total cost and revenue from the production and sale of x items. Find each of the following:

a. Total profit , $P(x)$.

b. Total cost, revenue, and profit from the production and sale of 50 units of the product.

c. The marginal cost, revenue, and profit when 50 units are produced and sold.

2.2.3 Application of Derivative in Road Taxation

Example

A motorist has to pay an annual road tax of 50 birrs and 110 birrs for insurance. His car does 30 miles to the gallon which costs 75 pence (per gallon). The car is serviced every 3000 miles for 20 birrs, and depreciation is calculated in pence by multiplying the square of the mileage by 0.001. Obtain an expression for the total annual cost of running the car. Hence find an expression for the average total cost per mile and calculate the annual mileage which will minimize the average cost per mile.

Solution

Suppose he covers x miles in a year.

Tax per annum = 50 birrs; Insurance per annum = 110 birrs;

Cost of petrol $=\dfrac{0.75x}{30}$; service charge $=\dfrac{20x}{3000}$; depreciation $=\dfrac{0.001x^2}{100}$ (0.001 is in pence and is divided by 100 to get the amount in birr).

Total cost (TC): $C = 50+110+\left(\dfrac{0.75x}{30}\right)+\dfrac{20x}{3000}+0.00001x^2$

Average total cost (TC) per mile: $\dfrac{C}{x} = M = \dfrac{160}{x}+\dfrac{0.75}{30}+\dfrac{20}{3000}+0.00001x$

$$\frac{dM}{dx} = 0$$

$$\Rightarrow \frac{-160}{x^2}+0.00001 = 0$$

$$\Leftrightarrow 0.00001x^2 = 160$$

$$\Leftrightarrow x^2 = \frac{160}{0.00001} = 16{,}000{,}000$$

$$\Leftrightarrow x = \pm 4{,}000$$

Now, to check whether this critical number $(x = 4000)$ gives a maximum or minimum cost, we can use a sign chart and apply the first derivative test.

		-4000		4000	
$x-4000$	----		-----	0	++++
$x+4000$	----	0	++++		++++
$x^2-16{,}000{,}000$	+++		-----		++++

From the sign chart, we see that the sign of the derivative changes from negative to positive at $x = 4000$. So, the motorist can cover 4000 miles in a year to minimize the average cost per mile.

Exercise 2.20

1. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing 1 dollar per foot and the remaining sides will use fencing costing 2 dollars per foot. Find the dimensions of the rectangle of the least cost.
2. A manufacturer wants to design an open box that has a square base and a surface area of 48 square unit. What dimensions will produce a box with a maximum volume?
3. A closed cylindrical can having a volume of 128π cm^3 is to be formed. Find the dimensions (the radius and height) of the cylinder that will minimize the amount of material to be used?

2.3 Introduction to Integration

Activity 2.17

1. Is it possible to determine the distance a vehicle has traveled if we know its velocity function?
2. Can we determine a company's total profit if we know its marginal-profit function?
3. An object moves with a velocity of $v(t)=\frac{1}{2}t$, where t is in minutes and v is in feet per minute.

   a) How far does the object travel during the first 30 min?

   b) How far does the object travel between the first hour and the second hour?
4. Find the areas of the region below $y = 2x$ and above the x-axis shown in the figure 2.16 (a) below.

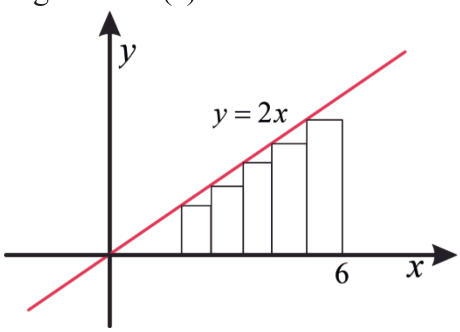


Figure 2.16 (a)

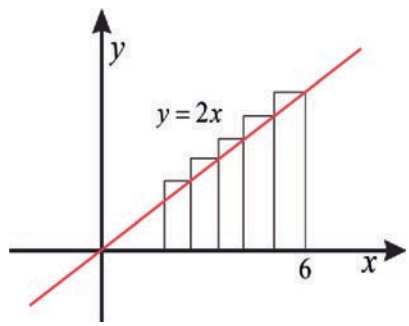


Figure 2.16 (b)

5. Find the area of the region below the line $y = 2x$ shown in the fig 2.16 (b). Upper sum on the interval $[0,6]$, with equal partition of $[0,6]$.
6. Find the area of the region just below the line $y = 2x$ over the interval $[0,6]$.)
7. Compare the values of 4 up to 6 above.

2.3.1 Areas of Regions

Integral calculus is primarily concerned with the area below the graph of a function (specifically, the area between the graph of a function and the x -axis). In this section, we assume that all functions are non-negative; that is, $f(x) \geq 0$.

Example 1

A vehicle travels 50 mi/hr for 2 hr. How far has the vehicle traveled?

Solution

The answer is 100 miles. We treat the vehicle's velocity as a function, $v(x) = 50$. We graph this function, sketch a vertical line at $x = 2$, and obtain a rectangle. This rectangle measures 2 units horizontally and 50 units vertically. Its area is the distance the vehicle has traveled:

$v(t) = 50$

Distance $A = 2 \times 50 = 100$

$2 \text{ hr} \times \dfrac{50 \text{ miles}}{1 \text{ hr}} = 100 \text{ miles}.$

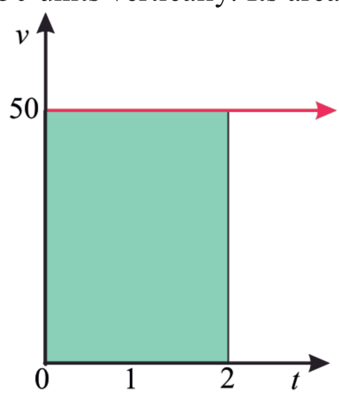


Figure 2.17

Example 2

Green Leaf Skateboards determines that for the first 50 skateboards produced, its cost is 40 birr per skateboard. What is the total cost to produce 50 skateboards?

Solution

The marginal-cost function is $C'(x) = 40,\ 0 \leq x \leq 50$. Its graph is a horizontal line.

If we mark off 50 units along the x-axis, we get a rectangle, as in Example 1.

The area of this rectangle is $40 \times 50 = 2000$. Therefore, the total cost to produce 50 skateboards is 2000 birr:

$$\left(50 \text{ skateboards } \times 40 \frac{\text{Birr}}{\text{skateboard}} = 2000 \text{ Birr}\right).$$

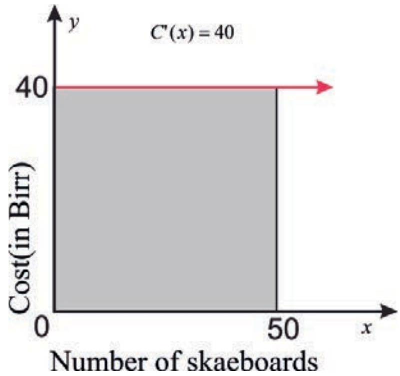


Figure 2.18

Example 3

The velocity of a moving object is given by the function $v(x) = 3x$, where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled during the first 3 hours $(0 \leq x \leq 3)$.

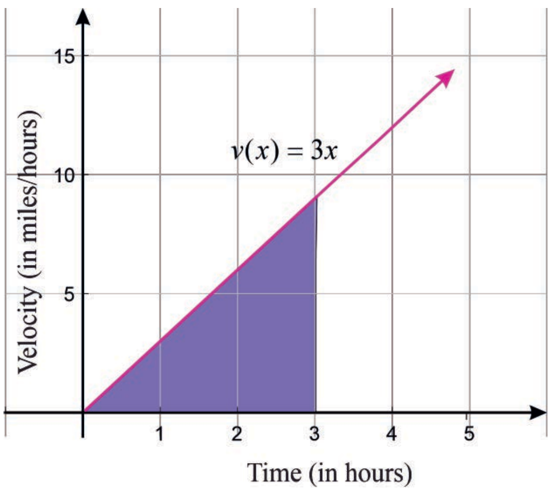


Figure 2.19

Solution

The graph of the velocity function is shown above. We see the region corresponding to the time interval $0 \le x \le 3$ is a triangle with base 3 and height 9 (since $v(3) = 9$).

Therefore, the area of this region is $A = \frac{1}{2}(3)(9) = \frac{27}{2} = 13.5.$

The object that traveled 13.5 miles during the first 3 hours.

In each of the Examples 1 through 3, the function was a rate function; its output units formed a rate (miles per hour in Examples1 and 3, Birr per skateboard in Example 2). The units of the area were derived by multiplying input units by output units.

Exercise 2.21

1. The velocity of a moving object is given by the function $v(x) = 3x$, where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled between the third hour and the fifth hour $(3 \le x \le 5)$.
2. A company has a marginal profit function modeled by the linear function $P'(x) = 0.15x$, where x is in months and P' is in thousands of dollars per month. Sketch this graph and use it to determine the total profit earned by the company in a year $(0 \le x \le 12)$.

Riemann summation

Activity 2.18

1. Examples 1 through 3 in this section suggest a pattern:

✓ If $f(x)=k$, where k is a non-negative constant, its graph is a horizontal line of height k. The region under this graph over the interval $[0, x]$ is a rectangle, and its area is $A = k.x$ (height times base).

✓ If $f(x)=mx$, its graph is a line of slope $m \geq 0$, passing through the origin. The region under this graph over an interval $[0, x]$ is a triangle, and its area is

$$A=\frac{1}{2}(x)(mx)=\frac{1}{2}mx^2.$$

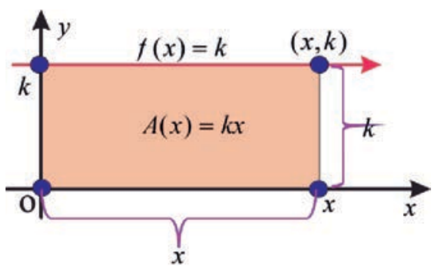


Figure 2.20

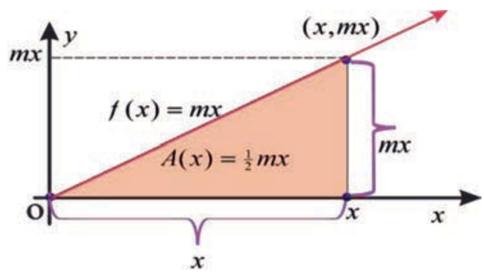


Figure 2.21

In these two cases, the area function is an anti-derivative of the function that generated the graph.

a. Is this always true?

b. Is the formula for the area under the graph of any function that function's anti-derivative?

c. How do we handle curved graphs for which area formulas may not be known?

2. Write summation notation for

i) $5+10+15+20+25$ ii) $g(x_1)\Delta x+g(x_2)\Delta x+\ldots+g(x_{19})\Delta x.$

Example 1

A Skateboard has the following marginal-cost function for producing skateboards: For up to 50 skateboards, the cost is 40 ETB per skateboard. For quantities from 51 through 125 skateboards, the cost drops to 30 ETB per skateboard. After 125 skateboards, it drops to 25 ETB per skateboard. If x represents the number of skateboards produced, the marginal cost function C' is

$$C'(x)=\begin{cases}40, & \text{for } 0\le x\le 50,\\ 30, & \text{for } 50<x\le 125,\\ 25, & \text{for } 125<x\le 150.\end{cases}$$

Find the total cost to produce 150 skateboards.

Solution

We are extending Example 2.We calculate the areas of the rectangles formed by the horizontal lines of the graph of the marginal-cost function:

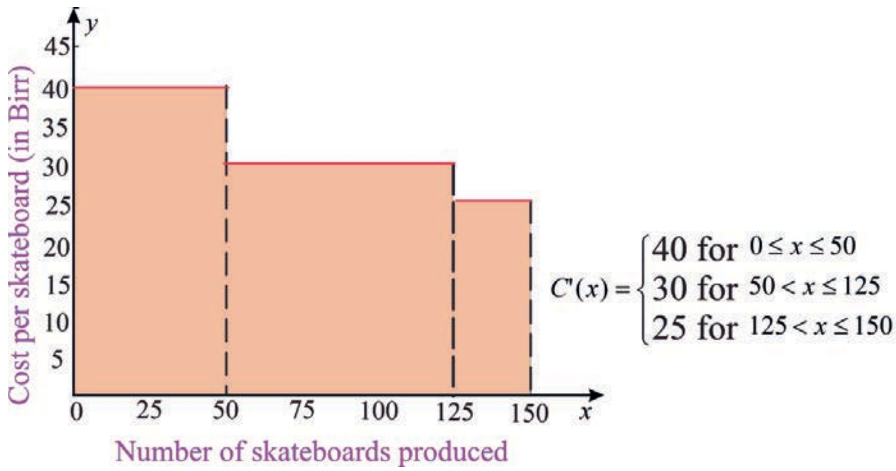


Figure 2.22

The total cost to produce 150 skateboards is found by summing those areas:

$$\text{Total cost} = (40)(50)+(30)(75)+(25)(25)=4875 \text{ Birr}.$$

Example 1 illustrates the first steps of a Riemann summation, a method that allows us to determine the area under a curved graph. We use rectangles to approximate the area under a curve given by $y = f(x)$, a continuous function, over an interval $[a, b]$. Riemann summation is accomplished with the use of summation notation, introduced below. In Figure 2.23, $[a, b]$ is divided into four subintervals, each having width $\Delta x = \frac{(b-a)}{4}$.

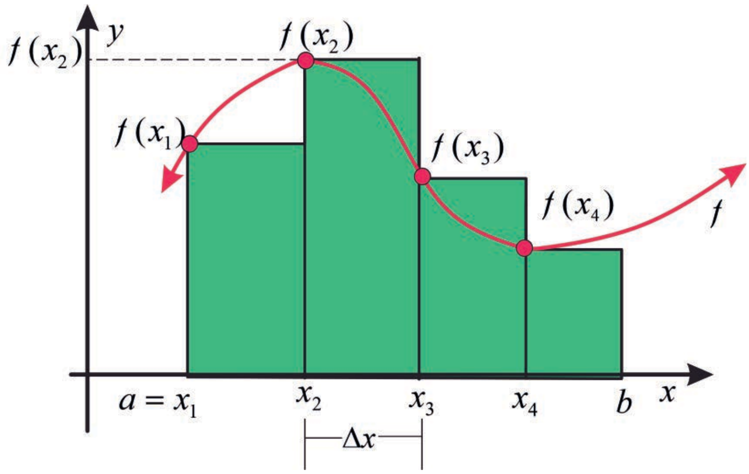


Figure 2.23

The heights of the rectangles shown are

$$f(x_1),\ f(x_2),\ f(x_3) \text{ and } f(x_4).$$

The area of the region under the curve is approximately the sum of the areas of the four rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

We can denote this sum with summation notation, or sigma notation, which uses the Greek capital letter sigma, Σ:

$$\sum_{i=1}^{4} f(x_i)\Delta x$$

This is read "the sum of the product $f(x_i)\Delta x$ from $i=1$ to $i=4$."

Before we continue, let's consider some examples involving summation notation.

Example 2

Find the sum of expression $\sum_{i=1}^{4} 3^i$.

Solution

$$\sum_{i=1}^{4} 3^i = 3^1 + 3^2 + 3^3 + 3^4 = 120.$$

Example 3

Express $\sum_{i=1}^{30} h(x_i)\Delta x$ without using summation notation.

Solution

$$\sum_{i=1}^{30} h(x_i)\Delta x = h(x_1)\Delta x + h(x_2)\Delta x + \ldots + h(x_{30})\Delta x.$$

Approximation of area by rectangles becomes more accurate as we use smaller subintervals and hence more rectangles, as shown in the following figures.

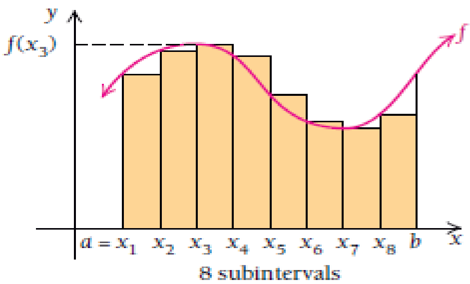


Figure 2.24a

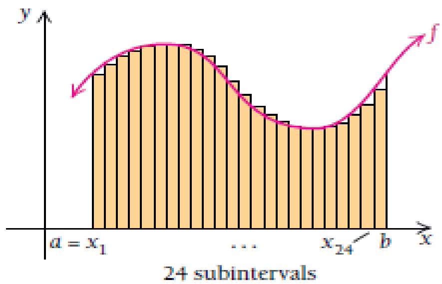


Figure 2.24b

In general, suppose that the interval $[a,b]$ is divided into n equally sized subintervals, each of width $\Delta x = \frac{(b-a)}{n}$ and we then construct rectangles with heights $f(x_1), f(x_2), \ldots, f(x_n)$.

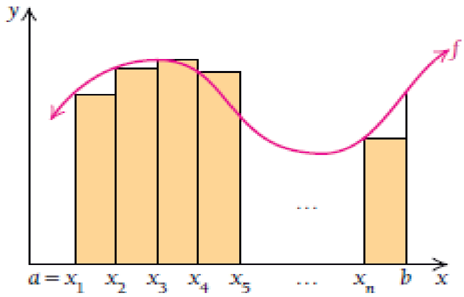


Figure 2.25

The width of each rectangle is Δx, so the first rectangle has an area of $f(x_1)\Delta x$, the second rectangle has an area of $f(x_2)\Delta x$, and so on. The area of the region under the curve is approximated by the sum of the areas of the rectangles A_n:

$$\text{Area}(A_n) = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Definition 2.12

Let $p = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition on $[a, b]$ of a given function *f*. For each k between 1 and n, let t_k be an arbitrary number in $[x_{k-1}, x_k]$. Then the sum

$$f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + f(t_3)\Delta x_3 + \ldots + f(t_n)\Delta x_n$$

is called **a Riemann sum for** f on $[a, b]$ and is denoted

$$\sum_{k=1}^{n} f(t_k)\Delta x_k.$$

Thus, $\sum_{k=1}^{n} f(t_k)\Delta x_k = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + f(t_3)\Delta x_3 + \ldots + f(t_n)\Delta x_n$

Riemann sum (pronounced "Ree-mahn") was developed by the great German mathematician G. F. Bernhard Riemann (1826–1866). Riemann sums can be calculated using any x-value within each subinterval. For simplicity, in this text, we will always use the left endpoint of each subinterval.

Steps for the Process of Riemann Summation

1. Draw the graph of $f(x)$.
2. Subdivide the interval $[a, b]$ into n subintervals of equal width. Calculate the width of each rectangle by using the formula $\Delta x = \frac{b-a}{n}$.
3. Construct rectangles above the subintervals such that the top left corner of each rectangle touches the graph.
4. Determine the area of each rectangle.
5. Sum these areas to arrive at an approximation for the total area under the curve.

Exercise 2.22

1. Find the approximate area of the region enclosed by the graph of function $f(x) = x^2 + 1$, the x-axis, $x = 1$ and $x = 5$.

   a. Use the following steps to find it.

Step 1: Interval $[1, 5]$ is divided into four subintervals, each having width

$\Delta x = \frac{5-1}{4} = 1$

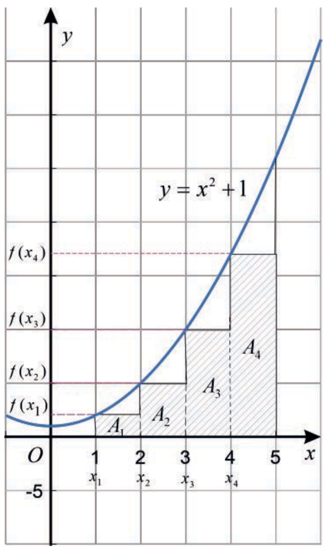


Figure 2.26

Step 2: Find the values of $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$, when $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$.

Step 3: Since the area of the region is approximatily the sum of the areas of four rectangles A_1, A_2, A_3 and A_4, calculate $A_1 + A_2 + A_3 + A_4 = \sum_{i=1}^{4} f(x_i)\Delta x$.

b. Find the approximate area of the same region in the same way as a, when the interval $[1, 5]$ is divided into eight subintervals, each having width $\Delta x = \frac{5-1}{8} = 0.5$.

2. Express $\sum_{i=1}^{6} (i^2 + i)$ without using summation notation.

2.3.2 Definite Integrals

Activity 2.19

Let $f(x) = 3x^2$ defined on the interval $[0,5]$. Then, find the Riemann sum of f on $[0,5]$

a. By dividing the interval $[0,5]$ in to 5 sub-intervals of equal width.

b. By dividing the interval $[0,5]$ in to 10 sub-intervals of equal width.

The key concept developed in this section is that the more subintervals we use, the more accurate the approximation of area becomes. As the number of subdivisions $dx = \frac{b-a}{n}$ increases, the width of each rectangle Δx decreases. If n is allowed to approach infinity, then Δx approaches 0; these are limits, and the approximations of the area becomes closer and closer to the true area under the graph. The exact area underneath the graph of a continuous function $y = f(x)$ over an interval $[a,b]$ is, by definition, given by a definite integral.

Definition 2.13

Let a given piecewice contionuous function $y = f(x)$ be non-negative, $f(x) \geq 0$, over an interval $[a,b]$. **A definite integral** is the Riemann sum of the areas of the rectangles under the graph of the function $y = f(x)$ over the interval $[a,b]$ as the value of n increases indefinitely (equivalently, $dx = \frac{b-a}{n}$ approaches very close to 0). That is:

$$\text{Exact area} = \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx, \text{ as } \Delta x \text{ gets very close to } 0.$$

Notice that the summation symbol becomes an integral sign (the elongated "s" is Leibniz notation representing "sum") and Δx becomes dx. If $f(x) \geq 0$ over an interval $[a,b]$, the definite integral represents the area.

Example

Determine the value of $\int_0^2 (3x+2)dx$.

Solution

This definite integral is a command to calculate the exact area underneath the graph of the function $f(x) = 3x + 2$ over the interval $[0,2]$.

We sketch the graph and note that the region is a trapezoid. Thus, we can use geometry to determine this area (a Riemann sum is not needed here).

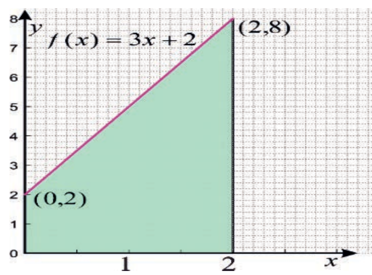


Figure 2.27

We find that the area is 10. Therefore, $\int_0^2 (3x+2)dx = 10$.

Exercise 2.23

Derermine the values of the following definite integrals:

a. $\int_0^2 (2x+1)dx$ b. $\int_0^4 2xdx$ c. $\int_1^5 xdx$ d. $\int_{-2}^3 (x+2)dx$

Anti-derivative

Activity 2.20

1. Find at least three different functions which have the derivative $2x$. Describe the similarities (and differences) between the functions.
2. Check that all of these functions:

   $f(x)=x^2+3x+1$, $g(x)=x^2+3x$, $h(x)=x^2+3x+12$, and $k(x)=x^2+3x+\pi$

   have the same derivative; $2x+3$.

Definition 2.14

The process of finding $f(x)$ from its derivative $f'(x)$ is known as anti - differentiation or integration. $f(x)$ is said to be the **anti-derivative** of $f'(x)$.

$$f(x) \xrightarrow{\text{differentiation}} f'(x)$$

$$f(x) \xleftarrow[\text{Anti-differentiation}]{} f'(x)$$

Integration is the reverse operation of differentiation.

- The set of all anti-derivatives of a function $f(x)$ is called the **indefinite integral** of $f(x)$.
- The indefinite integral of $f(x)$ is denoted by $\int f(x)dx$ read as "the integral of $f(x)$ with respect to x".
  - ✓ The symbol $\int$ is said to be the **integral sign**.
  - ✓ dx denotes that the variable of integration is x.
  - ✓ If a function has an integral, then it is said to be integrable.
  - ✓ If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + c$.
  - ✓ c is known as the constant of integration.

In this section, you will see how to find the integrals of constant and power functions.

i. Integration of a power function

Example 1

$\int x\,dx = \frac{1}{2}x^2 + c$ Because $\frac{d}{dx}\left(\frac{1}{2}x^2 + c\right) = \frac{1}{2} \times 2x + 0 = x$

$\int x^2\,dx = \frac{1}{3}x^3 + c$ Because $\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = \frac{1}{3} \times 3x^2 + 0 = x^2$

$\int x^3\,dx = \frac{1}{4}x^4 + c$ Because $\frac{d}{dx}\left(\frac{1}{4}x^4 + c\right) = \frac{1}{4} \times 4x^3 + 0 = x^3$

$\int 1\,dx = x + c$ Because $\frac{d}{dx}(x + c) = 1 + 0 = 1$

In general, the following formula holds. However, α is a real number and $\alpha \neq -1$, $\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1} + c$ (c is the constant of integration)

Note: Since $1 = x^0$, $\int 1\,dx = \int x^0\,dx = \frac{1}{0+1}x^{0+1} + c = x + c$

Example 2

Find the following indefinite integrals:

a. $\int x^7\, dx$ b. $\int x^{-3}\, dx$ c. $\int x^{\frac{1}{2}}\, dx$ d. $\int x^{-\frac{4}{3}}\, dx$

Solution

a. $\int x^7\, dx = \frac{1}{7+1}x^{7+1} + c = \frac{1}{8}x^8 + c.$

b. $\int x^{-3}\, dx = \frac{1}{-3+1}x^{-3+1} + c = -\frac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c.$

c. $\int x^{\frac{1}{2}}\, dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c = \frac{2}{3}x\sqrt{x} + c.$

d. $\int x^{-\frac{4}{3}}\, dx = \frac{1}{-\frac{4}{3}+1}x^{-\frac{4}{3}+1} + c = \frac{5}{2}x^{-\frac{1}{3}} + c = \frac{5}{2\sqrt[3]{x}} + c.$

Note: c is the constant of integration.

Exercise 2.24

Find the following indefinite integrals:

a. $\int x^5\, dx$ b. $\int x^{-4}\, dx$ c. $\int x^{\frac{1}{3}}\, dx$ d. $\int x^{-\frac{3}{2}}\, dx$

e. $\int x^{10}\, dx$ f. $\int x^{-2}\, dx$ g. $\int x^{\frac{3}{2}}\, dx$ h. $\int x^{-\frac{3}{5}}\, dx$

ii. Properties of the indefinite integral

1. Let k be a constant, $\int kf(x)dx = k \int f(x)dx$
2. $\int\big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$
3. $\int\big(f(x) - g(x)\big)dx = \int f(x)dx - \int g(x)dx$

Note

$\int k\ dx = kx + c$, where k is a given constant and c is the constant of integration.

Example

Find the following indefinite integrals:

a. $\int 10\, dx$ b. $\int (2x^5 - x)\, dx$ c. $\int \left(x^2 + \sqrt{x}\right) dx$

Solution

$a. \int 10\, dx = 10x + c$

$b. \int (2x^5 - x)\, dx = 2 \cdot \frac{1}{6}x^6 - \frac{1}{2}x^2 + c = \frac{1}{3}x^6 - \frac{1}{2}x^2 + c$

$c. \int \left(x^2 + \sqrt{x}\right) dx = \frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + c$

Note: Add only one constant of integration c at the end.

Exercise 2.25

Find the following indefinite integrals.

a. $\int 99\, dx$ b. $\int 10x^4\, dx$ c. $\int \frac{3}{x} dx$ d. $\int (x^2 - x)\, dx$

e. $\int (3x^2 - 2x + 5)\, dx$ f. $\int (-2x^2 - x - 6)\, dx$ g. $\int \left(x^{-2} - 4x^3 + \sqrt{x}\right) dx$

2.3.3 Area and Definite Integrals

In Sections 2.3.1 and 2.3.2, we considered the area under the graph of a function f. We have yet to establish the general rule to find the exact area under the graph of f. As we will see, we can use the anti-derivative of a function to determine the exact area under the graph of the function. This process is called integration.

Definition 2.15

Let f be a given function over the interval $[a, b]$ and F be any anti-derivative of f. Then the definite integral of f from a to b is: $\int_a^b f(x)dx = F(b) - F(a)$.

Evaluating definite integrals is called *integration*. The numbers a and b are known as *the limits of integration*.

Theorem 2.7 (The Fundamental Theorem of Calculus)

Let f be a given non-negative function over an interval $[0,b]$, and let $A(x)$ be the area between the graph of f and the x-axis over the interval $[0,x]$, with $0<x<b$. Then $A(x)$ is a differentiable function of x and $A'(x)=f(x)$.

Note

To find the area under the graph of a nonnegative function f over the interval $[a,b]$:

1. Find any anti-derivative $F(x)$ of $f(x)$. Let $C=0$ for simplicity.
2. Evaluate $F(x)$ at $x=b$ and $x=a$, and compute $F(b)-F(a)$.

   The result is the area under the graph over the interval $[a,b]$.

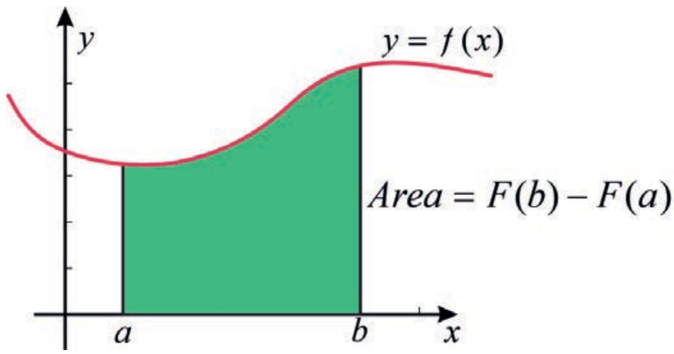


Figure 2.28

Example 1

Find the area under the graph of $y=x^2+1$ over the interval $[-1,2]$.

Solution

In this case, $f(x)=x^2+1$, with $a=-1$ and $b=2$.

1. Find any anti-derivative of $x=b$ of $f(x)$. We choose the simplest one:

$$F(x)=\frac{x^3}{3}+x.$$

2. Substitute 2 and -1, and find the difference $F(2)-F(-1)$:

$$\int_{-1}^{2}(x^2+1)dx=\left[\frac{x^3}{3}+x\right]_{-1}^{2}$$

$$=\left[\frac{2^3}{3}+2\right]-\left[\frac{(-1)^3}{3}+(-1)\right]$$

$$=6$$

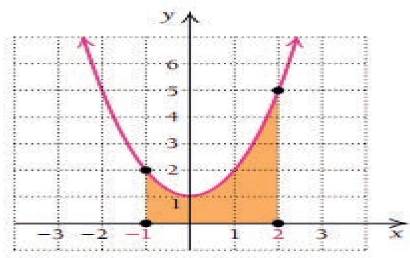


Figure 2.29

We can make a partial check by counting the squares and parts of squares shaded on the graph in Figure 2.29.

Exercise 2.26

1. Refer to the function and graph in Example 1.
   a. Calculate the area over the interval $[0,5]$.
   b. Calculate the area over the interval $[-2,2]$.
   c. Can you suggest a shortcut for part (b)?
2. Evaluate each of these integrals.

a. $\int_{1}^{3}(3x^2+2x)dx$ b. $\int_{-1}^{0}(x^3-3x+1)dx$ c. $\int_{4}^{15}(0.002x^4-0.3x^2+4x-7)dx$

Example 2

Let $y = x^3$ represent the number of kilowatts (kW) generated by a new power plant each day, x days after going online. Find the area under the graph of $y = x^3$ over the interval $[0, 5]$ and interpret the significance of the area.

Solution

In this case, $f(x) = x^3, a = 0,$ and $b = 5$.

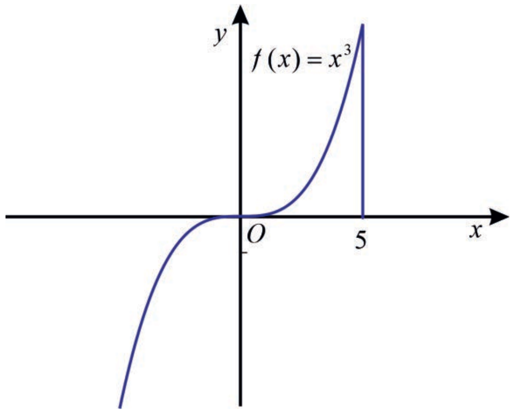


Figure 2.30

1. Find any anti-derivative $F(x)$ of $f(x)$. We choose the simplest one:

$$F(x) = \frac{x^4}{4}$$

2. Substitute 5 and 0, and find the difference $F(5) - F(0)$:

$$F(5) - F(0) = \frac{5^4}{4} - \frac{0^4}{4} = \frac{625}{4} = 156\frac{1}{4}.$$

The area represents the total number of kilowatts generated during the first 5 days.

The difference $F(b) - F(a)$ has the same value for all anti-derivatives F of a function $f(x)$ whether the function is nonnegative or not. It is called the definite integral of $f(x)$ from a to b.

Properties of definite integrals

If f and g are given function on $[a,b], k \in \Re$, and $c \in [a,b]$, then

a. $\int_a^a f(x)dx = 0$

b. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

c. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

d. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

e. $\int_a^b (f(x) \pm g(x))dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$

Example 3

Evaluate $\int_a^b x^2 dx.$

Solution

Using the anti-derivative $F(x) = \frac{x^3}{3}$, we have $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$

It is convenient to use an intermediate notation: $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$

where $F(x)$ is an anti-derivative of $f(x)$.

Example 4

Evaluate $\int_{-1}^4 (x^2 - x)dx$

Solution

$a)$ $\int_{-1}^4 (x^2 - x)dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^4$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$= \frac{64}{3} - 8 + \frac{1}{3} + \frac{1}{2} = 14\frac{1}{6}.$$

Exercise 2.27

Evaluate the following definite integrals:

a. $\int_1^3 x^2 dx$ b. $\int_0^2 (x^2 - 2x + 8)dx$ c. $\int_{-2}^3 (-3x^2 + 4x - 5)dx$ d. $\int_1^4 \left(2x + \frac{1}{x^2}\right)dx$

2.3.4 More on Area

When we evaluate the definite integral of a nonnegative function, we get the area under the graph over an interval.

Activity 2.21

Find the areas of a region bounded by the function $f(x)$ and the interval $[a,b]$, where a. $f(x) = k,\ k > 0$. b. $f(x) = x$. c. $f(x) = x^2$.

Example 1

Find the area under $y = \frac{1}{x^2}$ over the interval $[1, b]$.

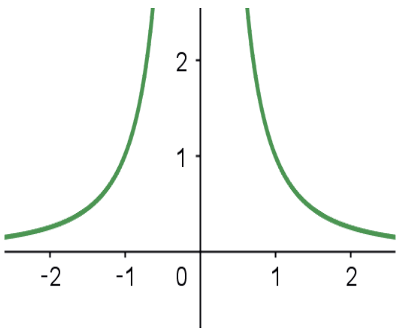


Figure 2.31

Solution

$$\int_1^b \frac{dx}{x^2} = \int_1^b x^{-2} dx = \left[\frac{x^{-2+1}}{-2+1}\right]_1^b = \left[\frac{x^{-1}}{-1}\right]_1^b = \left[-\frac{1}{x}\right]_1^b = \left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right) = 1 - \frac{1}{b}.$$

The definite integral is also defined for $f(x) < 0$. We can use geometry to determine the value of some definite integrals, as the following example suggests.

Example 2

Compare the definite integrals of the functions $y = x^2$ and $y = -x^2$

Solution

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

and

$$\int_0^2 -x^2 dx = \left[-\frac{x^3}{3}\right]_0^2 = -\frac{2^3}{3} - \frac{0^3}{3} = -\frac{8}{3}.$$

The graphs of the functions $y = x^2$ and $y = -x^2$ are reflections of each other across the x-axis (see Figure2.32a and Figure2.32b). Thus, the shaded areas are the same, $\frac{8}{3}$. The evaluation procedure for $y = -x^2$ gave us $-\frac{8}{3}$. This illustrates that for negative-valued functions, the definite integral gives us the opposite of the area between the curve and the x-axis.

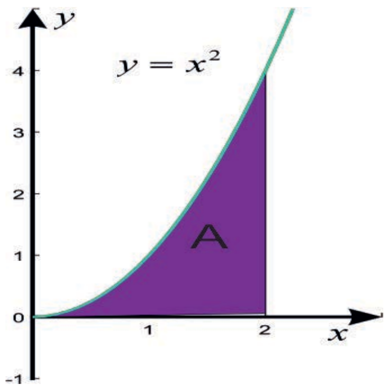


Figure 2.32a

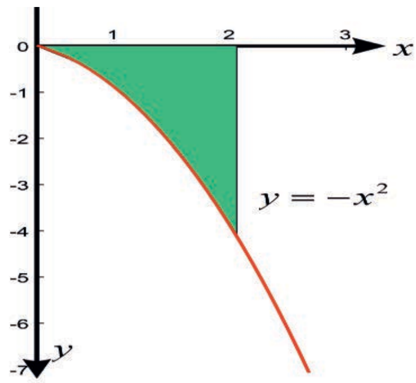


Figure 2.32b

When $f(x) \geq 0$ in the interval $[a,b]$, the area S which is enclosed by the graph of $f(x)$, the two straight lines $x = a, x = b$ and the x-axis is $S = \int_a^b f(x)dx$

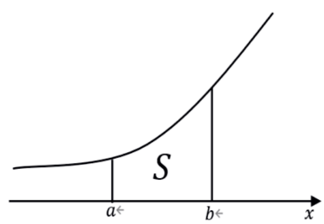


Figure 2.33

When $f(x) \le 0$ in the interval $[a, b]$, the areas S which is enclosed by the graph of $f(x)$, the two straight lines $x = a, x = b$, and the x-axis is

$$S = -\int_a^b f(x)dx$$

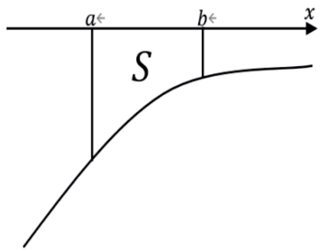


Figure 2.34

Example 3

Find the area S which is enclosed by the graph of $f(x)$and the x-axis.

a. $f(x) = -x^2 + 1$ b. $f(x) = x^2 - 1$

Solution

a. From the figure on the right, the area S is

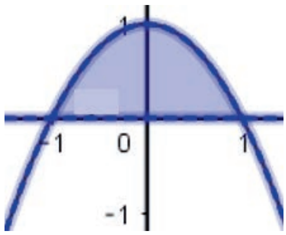


Figure 2.35

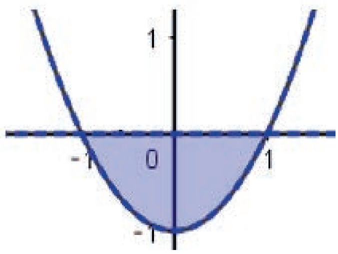
$$S = \int_{-1}^{1}(-x^2 + 1)\,dx$$

$$= \left[-\frac{x^3}{3} + x\right]_{-1}^{1}$$

$$= \left(-\frac{1^3}{3} + 1\right) - \left\{-\frac{(-1)^3}{3} + (-1)\right\}$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

a. From the figure on the right, the area S is



$$S = -\int_{-1}^{1}(x^2 - 1)\,dx$$

$$= -\left[\frac{x^3}{3} - x\right]_{-1}^{1}$$

$$= -\left[\left(\frac{1^3}{3} - 1\right) - \left\{\frac{(-1)^3}{3} - (-1)\right\}\right] = -\left(-\frac{2}{3} - \frac{2}{3}\right) = \frac{4}{3}$$

Figure 2.36

When the area S is divided into the upper side and the lower side of the x-axis, the upper area and the lower side area are obtained respectively and added them.

In the figure below, the area S of the area enclosed by the graph of $f(x)$ and the x-axis can be found by $S = A + B$.

In the interval $[a, b]$, $f(x) \leq 0$. In the interval $[b, c]$, $f(x) \geq 0$.

So, we can find the area S,

$$S = -\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

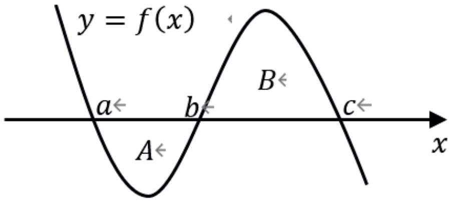


Figure 2.37

Example 4

Find the area S of the region enclosed between the graph of $f(x) = x^2 - 1$, the x-axis, and the y-axis in the interval $[0, 2]$.

Solution

Note that 1 is the x-intercept of the function $f(x) = x^2 - 1$ in $[0, 2]$.

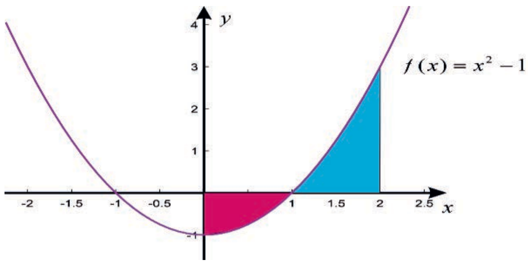


Figure 2.38

$$S = \int_0^2 (x^2 - 1)dx = -\int_0^1 (x^2 - 1)dx + \int_1^2 (x^2 - 1)dx$$

$$= -\left[\frac{x^3}{3} - x\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2$$

$$= -\left[\left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)\right] + \left[\left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right)\right]$$

$$= -\left(-\frac{2}{3}\right) + \frac{4}{3} = \frac{2}{3} + \frac{4}{3} = 2$$

This shows that the area above the x-axis exceeds the area below the x-axis by 2 unit.

So, the area of the region is 2 units.

The definite integral of a continuous function over an interval is the sum of the areas above the x-axis minus the sum of the areas below the x-axis.

Example 5

Consider $\int_{-1}^{2}(-x^3+3x-1)dx$. Predict the sign of the result by examining the graph, and then evaluate the integral.

Solution

From the graph, it appears that there is considerably more area below the x-axis than above. Thus, we expect that

$$\int_{-1}^{2}(-x^3+3x-1)dx<0.$$

Evaluating the integral, we have

$$\int_{-1}^{2}(-x^3+3x-1)dx=\left[-\frac{x^4}{4}+\frac{3}{2}x^2-x\right]_{-1}^{2}$$

$$=\left(-\frac{2^4}{4}+\frac{3}{2}.2^2-2\right)-\left(-\frac{(-1)^4}{4}+\frac{3}{2}.(-1)^2-(-1)\right)$$

$$=(-4+6-2)-\left(-\frac{1}{4}+\frac{3}{2}+1\right)$$

$$=0-2\frac{1}{4}=-2\frac{1}{4}.$$

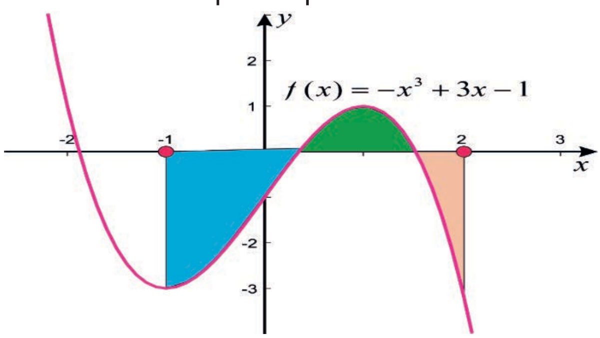


Figure 2.39

As a partial check, we note that the result is negative, as predicted.

Exercise 2.28

1. Find the area S which is enclosed by the graph of $f(x)$ and the x-axis.

   a. $f(x) = x^2 - 4$ b. $f(x) = -x^2 + 4$

   c. $f(x) = x^2 - 2x$ d. $f(x) = -x^2 - 2x$

2. Find the area S which is enclosed by the graph of $f(x)$, the two straight lines and the x-axis.

   a. $f(x) = x^2$, the two straight lines $x = 0$ and $x = 4$

   b. $f(x) = x^2 + 3$, the two straight lines $x = -1$ and $x = 3$

3. Find the area S which is enclosed between the graph of

   $f(x) = x(x+2)(x-1)$ and the x-axis.

2.3.5 Application of Integration

Activity 2.22

Do you think that integration has any other applications than finding the area under the curve of the function? Write down what you think these applications might be.

In this section, we will look at some applications of integrals. Integration is used in different fields of study such as business, physical sciences, economics, etc.

Example 1

A certain Airlines determines that the marginal profit resulting from the sale of x seats on a plane traveling from Addis Ababa to New York City, in hundreds of dollars, is given by $P'(x) = \sqrt{x} - 6$.

Find the total profit when 60 seats are sold.

Solution

We integrate to find $P(60)$:

$$P(60) = \int_0^{60} P'(x)dx = \int_0^{60} (\sqrt{x} - 6)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - 6x\right]_0^{60} \approx -50.1613.$$

When 60 seats are sold, the airline's profit is -$5016.13. That is, the airline will lose $5016.13 on the flight.

Example 2

Suppose that the velocity function of a car is $v(t) = 5t^4$ and that the displacement at $t = 0$ is $s(0) = 9$. Find $s(t)$. Assume that $s(t)$ is in meters and $v(t)$ is in meter per second.

Solution

First $s(t) = \int v(t)dt = \int 5t^4 dt = t^5 + C.$

Next, we determine C by using the initial condition $s(0) = 9$, which is the displacement of the car at time $t = 0$: $s(0) = 0^5 + C = 9 \Rightarrow C = 9$ Thus, $s(t) = t^5 = 9$.

Example 3

The driver of a vehicle traveling at 40 mi/ hr (58.67 ft /sec) applies the brakes, softly at first, then harder, coming to a complete stop after 7 seconds. The velocity as a function of time is modeled by $v(t) = -1.197t^2 + 58.67$, where v is in feet per second, t is in seconds, and $0 \le t \le 7$. How far did the vehicle travel while the driver was braking?

Solution

The distance traveled $s(t)$ is given by the definite integral of $v(t)$:

$$s(t)=\int_0^7(-1.197t^2+58.67)dt=\left[-\frac{1.197}{3}t^3+58.67t\right]_0^7$$

$$=-\frac{1.197}{3}(7)^3+58.67(7)-0$$

$$\approx 273.83 \text{ ft}$$

Exercise 2.29

1. Pure Water Enterprises finds that the marginal profit, in dollars, from drilling a well that is x feet deep is given by $P'(t)=\sqrt[5]{x}$. Find the profit when a well 250 ft deep is drilled.
2. A company estimates that its sales will grow continuously at a rate given by the function $S'(t)=20e^t$, where $S'(t)$ is the rate at which sales are increasing, in dollars per day, on day t.
   a. Find the accumulated sales for the first 5 days.
   b. Find the sales from the 2^{nd} day through the 5^{th} day.

Summary

- **Rate of change, gradient, and derivative of functions**
  - A **rate of change** is a rate that describes how one quantity changes in relation to another quantity.
  - If $y = f(x)$ then the **average rate of change** of y with respect to x on the interval $[a, b]$ is
    $$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$
  - A **secant line** is a line that intersects the graph of the function $y = f(x)$ at exactly two points.
  - A **tangent line** to a curve or a graph of a function $y = f(x)$ is a line that touches the curve exactly at one point. The point where the tangent line touches the graph is said to be the point of tangency.
  - A **gradient** is a measure of steepness and direction of a line.
    $$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x}.$$
  - The derivative of a function $f(x)$ at a number a in the domain of f, denoted by $f'(a)$ is the gradient of a tangent line to the graph of $f(x)$ at $(a, f(a))$.
    That is as h gets very close to zero from both sides, $f'(a) = \frac{f(a+h) - f(a)}{h}$.

- **The derivatives of some functions**
  1. If $f(x) = k$, then $f'(x) = 0$.
  2. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
  3. If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$.

- **Derivatives of combinations of functions**

If f and g are differentiable functions, then $f+g$, kf, fg and $\frac{f}{g}$ are also differentiable and are given as follows:

i. $(f+g)'(x) = f'(x) + g'(x)$.

ii. $(kf)'(x) = kf'(x)$, where k is a constant number.

iii. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

iv. $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

- **Differentiations of the composition of functions**

✓ Let g be differentiable at x, and f be differentiable at $g(x)$ .Then fog is differentiable at x, and $(\text{fog})'(x) = f'(g(x))g'(x)$ for all x such that g is differentiable at x and f is differentiable at $g(x)$.

✓ If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

- **Equation of tangent line and normal line to a curve**

a. The equation of the tangent line to the curve $y = f(x)$ at $x = a$ is given by

$y = f'(a)(x-a) + f(a)$.

b. The equation of the normal line to the curve $y = f(x)$ at $x = a$ and $f'(x) \neq 0$ is given by

$$y = \frac{-1}{f'(a)}(x-a) + f(a).$$

- A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is horizontal or for which the derivative does not exist. That is, c is a critical value if $f(c)$ exists and $f'(c) = 0$ or $f'(c)$ does not exist.

- Let I be the domain of f. $f(c)$ is a **relative minimum** if there exists within I an open interval I_1 containing c such that $f(c) \leq f(x)$, for all x in I_1 and is a **relative maximum** if there exists within I an open interval I_2 containing c such that $f(c) \geq f(x)$, for all x in I_2.
- **The First-Derivative Test for Relative Extrema**

  Let f be a function that is differentiable on an interval I. Then

  i. If f' changes sign from positive to negative at c then f has **a local maximum** value at c for some critical value c.

  ii. If f' changes sign from negative to positive at c then f has **a local minimum** value at c for some critical value c.
- The **anti-derivative** of $f(x)$ is the set of functions $F(x)+C$ such that $\frac{d}{dx}(F(x)+C)=f(x)$, where the constant C is called the **constant of integration.**
- Let f be continuous on $[a,b]$, and let $p=\{x_0,x_1,x_2,\ldots,x_n\}$ be any portion of $[a,b]$. Then the sum

  $\sum_{k=1}^{n} f(t_k)\Delta x_k = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + f(t_3)\Delta x_3 + \ldots + f(t_n)\Delta x_n$ is called a **Riemann sum for f** on $[a,b]$.
- Fundamental Theorem of Calculus:

  If $f(x)=F'(x)$, then $\int_a^b f(x)dx = F(b)-F(a)$.
- If a function has areas both below and above the $x-$axis, the definite integral gives the net total area or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.
  - ✓ If there is more area above the x-axis than below, the definite integral will be positive.

✓ If there is more area below the x-axis than above, the definite integral will be negative.

✓ If the areas above and below the x-axis are the same, the definite integral will be 0.

- Properties of definite integrals

a. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

b. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

c. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

d. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

e. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Review Exercise

1. Find the derivative of each of the following functions at the given number.

a. $f(x)=\dfrac{x^3+3x}{x^2}$ at $x=2$

b. $f(x)=\dfrac{5}{x^3+2}$ at $x=-4$

c. $f(x)=\dfrac{x^4}{x^3+2}$ at $x=2$

d. $f(x)=\dfrac{4\sqrt{x}}{2x+1}$ at $x=4$

2. The diagram in Figure 2.40 shows the graph of the function $y=\dfrac{1}{x}$ between $x=\dfrac{1}{2}$ and $x=4$.

i. Find the slope of the secant line between

a. $x=3$ and $x=3.1$.

b. $x=3$ and $x=3.01$.

c. $x=3$ and $x=3.001$.

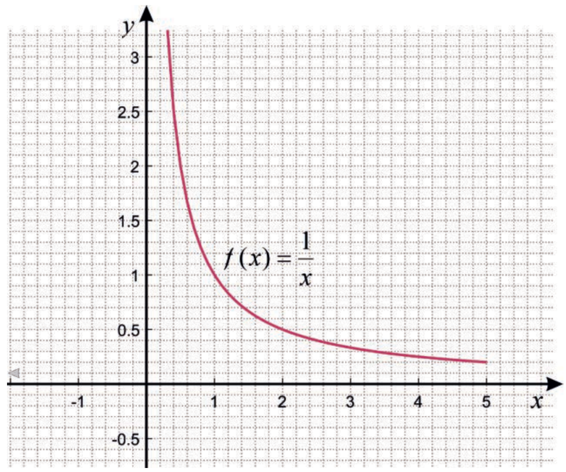


Figure 2.40

ii. Find a formula for the slope of the secant line between $(3, f(3))$ and $(3+\Delta x, f(3+\Delta x))$ for a function *f*.

iii. Determine what happens when Δx approaches 0.

3. If an object is dropped from an 80 m high window, its height y above the ground at the time t seconds is given by the formula $y=f(t)=80-4.9t^2$ (neglect air resistance).

i. Find the average velocity of the falling object between

a. $t=1$ sec and $t=1.1$ sec.

b. $t=1$ sec and $t=1.01$ sec.

c. $t=1$ sec and $t=1.001$ sec.

ii. Find a simple formula for the average velocity of the falling object between $t = 1$ sec and $t = (1 + \Delta t)$ sec.

iii. Determine what happens to this average velocity as Δt approaches 0.

4. Let $y = f(t) = t^2$ where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$.

   i. Make a table of the average speed of the falling object between

      a. $t = 2$ sec and $t = 3$ sec.

      b. $t = 2$ sec and $t = 2.1$ sec.

      c. $t = 2$ sec and $t = 2.01$ sec.

   ii. Find a simple formula for the average speed between time $t = 2$ sec and $t = (2 + \Delta t)$ sec.

   iii. In your formula for average speed determine what happens as Δt approaches zero.

   iv. Draw the straight line through the point $(2, 4)$ whose slope is the instantaneous velocity you just computed.

5. Find the equation of the tangent and the normal lines $f(x) = (1 - x^3)\sqrt{x + 2}$.

6. Find the derivative of each of the following functions.

   a. $f(x) = x^2(2x + 5)$

   b. $f(x) = \dfrac{x^2 + x}{x^3 - x + 2}$

   c. $f(x) = \dfrac{x^2}{x + 1}$

   d. $f(x) = x^3\left(1 - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$

7. Consider the function $f(x) = 6x^5 - 50x^3 - 120$.

   a. Find the interval on which f is increasing.

   b. Find the interval on which f is decreasing.

   c. Find all values of x for which f have a relative maximum.

   d. Find all values of x for which f have a relative minimum.

8. A particle moves along the curve $2y = x^2 + 4$. Find the point(s) on the curve at which the y-coordinate changes 4 times as fast as the x-coordinate.

9. Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on the interval [0, 2]. Is this value maximum or minimum?

10. Find the value of a, b, c, d so that $h(x) = ax^3 + bx^2 + cx + d$ will be extreme values at $(1, 2)$ and $(2, 3)$.

11. Find the absolute maximum and the minimum values of the following function on the interval [0,2].

$$f(x) = \begin{cases} x^3 - \frac{x}{3}; & 0 \le x \le 1 \\ x^2 + x - \frac{4}{3}; & 1 < x \le 2 \end{cases}$$

12. A board 5 ft long slides down a wall. At the instant when the bottom end is 4 ft from the wall, the other end is moving down at a rate of 2 ft/s. At that moment
    a. How fast is the bottom end sliding along the ground?
    b. How fast is the area of the region between the board, the ground, and the wall changing?

13. A farmer has 240 m of fencing material and wants to fence a rectangular field that borders a straight river. (No fence is needed along the river). What are the dimensions of the field that has the largest area?

14. The radius of the balloon is increasing at the rate of 0.5 cm/sec. At what rate the surface area of the balloon is increasing when the radius is 4 cm?

15. A cyclist decelerates at a constant rate from 30 km/hr to a standstill in 45 sec.
    a. How fast is the bicyclist traveling after 20 sec?
    b. How far has the bicyclist traveled after 45 sec?

16. A factory is polluting a lake in such a way that the rate of pollutants entering the lake at time t, in months, is given by $N'(t) = 280t^{\frac{3}{2}}$, where N is the total number of pounds of pollutants in the lake at time t.

Figure 2.41

a. How many pounds of pollutants enter the lake in 16 months?

b. An environmental board tells the factory that it must begin to clean up procedures after 50,000 pounds of pollutants has entered the lake. After what length of time will this occur?

UNIT 3

STATISTICS

Unit Outcomes

By the end of this unit, you will be able to:

- Describe absolute and relative dispersion and their interpretation.
- Conceptualize specific facts about measurement in statistical data.
- Grasp basic concepts about sampling techniques.
- Appreciate the value of statistics in real life.

Unit Contents

3.1 Measures of Absolute Dispersions

3.2 Relative Dispersions

3.3 Use of Frequency Curves

3.4 Sampling Techniques

Summary

Review Exercise

KEY Words

- Cluster sampling
- Quartile deviation
- Coefficient of mean deviation
- Random sampling technique
- Coefficient of quartile deviation
- Simple random sampling
- Coefficient of range
- Skewness
- Coefficient of variation
- Standard deviation
- Inter-quartile
- Stratified sampling
- Lower class boundary of the lowest class
- Symmetrical distribution
- Mean deviation
- Systematic sampling
- Multistage sampling
- Upper class boundary
- Non-random sampling technique
- Variation

Introduction

Statistics as a science deals with the proper collection, organization, presentation, analysis and interpretation of numerical data. Since statistics allows us to draw conclusions based on data it is applicable in almost all sciences. Some of the organizations where statistics is applicable are business, meteorology, schools, economic, social and political activities. In previous grades, you should have studied about statistics and its basic concepts such as collection of data, presentation of data, and measures of central tendencies like mean, median and mode and measures of dispersion such as range, variance, and standard deviation. In this unit, we shall revise those concepts and discuss additional measures of dispersion; namely, inter-quartile range, mean deviation, quartile deviation and standard deviations. In addition, we shall discuss some relative measures of dispersion such as the coefficient of dispersions. Finally, sampling techniques will be discussed.

3.1 Measures of Absolute Dispersions

In Grade 9, you learned about the different measures of dispersion. In this section, we shall revise measures of dispersion and discuss measure of absolute dispersion.

Activity 3.1

1. Why do you study dispersion?
2. What is the difference between measures of dispersion and measures of central tendency?

Measures of Dispersion

Recall that a measure of dispersion can be defined in either of the following ways:

a. The degree to which numerical data tends to spread about an average.

b. The scatter or variation of variables about a central value.

Types of measures of dispersion:

There are two types of measures of dispersion. These are:

i. Absolute measure of dispersion.

ii. Relative measure of dispersion.

i. Absolute measure of dispersion

The following are absolute measures of dispersions

- Range
- Inter-quartile range
- Mean deviation
- Standard deviation

Note

An absolute measure of dispersion measures the variability in terms of the same units of the data, for example if the units of the data are meters, liters, kg, etc. The units of the measures of dispersion will also be meters, liters, kg, etc.

ii. Relative measure of dispersion

The common relative measures of dispersion are:

- Coefficient of range
- Coefficient of quartile deviation
- Coefficient of mean deviation
- Coefficient of standard deviation or coefficient of variation

Note

A relative measure of dispersion compares the variability of two or more data that are independent of the units of measurement. It is used to compare the dispersion of a data set with the dispersion of another data set.

The major difference between absolute and relative measures of dispersion is that the absolute measure of dispersion measures only the variability of the data. Furthermore, it has the same unit of measurement. However, relative measure of dispersion is used to compare the variation of two or more distributions, and does not have units.

3.1.1 Range and Inter-quartile Range

Range

Activity 3.2

Consider the following data set which gives the price (in Birr) of a cup of coffee in randomly selected 20 coffee houses:

10, 5, 9, 12, 10, 12, 13, 8, 15, 12 , 12, 13, 10, 12, 5, 14, 6, 14, 10, 8.

a. Identify the highest price and lowest price of a cup of coffee.

b. Determine the difference between the highest price and the lowest price of a cup of coffee.

Range for ungrouped data

Definition 3.1

The range of an ungrouped set of data is defined as the difference between the largest value and smallest value in a set of data, that is, $R = L - S$, where L= largest value and S = smallest value.

Example 1

A certain supermarket has registered the following data about daily sales (in Birr 100) for twelve consecutive days:

$$115, 134, 120, 111, 122, 103, 80, 155, 122, 104, 110, 98.$$

Calculate the range of the data.

Solution

L = 155, S = 80, then R = L-S = $155 - 80 = 75$.

Range for grouped data

Definition 3.2

The range of a grouped set of data R is defined as the difference between the upper class boundary of the highest class $B_U(H)$, and the lower class boundary of the lowest class $B_L(L)$, that is, $R = B_U(H) - B_L(L)$.

Example 2

Consider the following data. What is the range of this distribution?

x	5-9	10-14	15-19	20-24
f	4	8	3	2

Solution

From the grouped frequency distribution, the range is calculated using

$$B_U(H) = 24.5, \qquad B_L(L) = 4.5.$$

Therefore, $R = B_U(H) - B_L(L) = 24.5 - 4.5 = 20.$

Advantages and Limitations of range

Advantage of range

- It is simple to compute.

Limitations of range

- It only depends on extreme values.
- It does not consider variations of values in between.
- It is highly affected by extreme values.

Exercise 3.1

1. Calculate the range of each of the following data sets:

   a. 25, 21, 22, 20, 19, 16

   b. 14, 17, 18, 16, 10, 18, 19, 22.

2. Calculate the range for these data sets.

   a.

x	5	7	8	9	11
f	13	10	11	12	13

   b.

x	0-20	20-40	40-60	60-80	80-100
f	3	1	2	4	1

Inter-Quartile Range

Activity 3.3

1. The following are the average daily temperatures (in°C) of ten days for two cities A and B:

   City A: 28, 24, 25, 26, 26, 20, 27, 30, 28, 27

   City B: 22, 24, 23, 26, 25, 25, 24, 26, 27, 28.

   a. Calculate the first quartile (Q_1) and the third quartile (Q_3) for each city.

   b. Determine $Q_3 - Q_1$ for each city.

2. Calculate Q_1 and Q_3 for the following data set:

x	0-5	5-10	10-15	15-20
f	1	3	3	2

In the above section, we mentioned range as the difference between the highest and the lowest values in a data set. Sometimes, it may not be possible to find the range, especially in open ended data where the highest or lowest value may be unknown. It may be the case that you know that the range is highly affected by extreme values. Under such circumstances, it may be of interest to measure the difference between the third quartile (Q_3) and the first quartile (Q_1) which is called the inter-quartile range. Inter-quartile range is a measure of variation which could possibly overcomes the limitations of range. It is defined as follows:

Definition 3.3

Inter-quartile range (IQR) is a difference between the upper and the lower quartiles; that is, $IQR = Q_3 - Q_1$.

Recall that the k^{th} $(k = 1, 2, 3)$ quartiles for ungrouped data are

a. odd $Q_k = \left(\frac{k(n+1)}{4}\right)^{th}$ item if n is odd,

b. even $Q_k = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th}$ item if n is even, where n is the number of observations of the data.

The $k^{th}(k = 1,2,3)$ quartiles for a grouped frequency distribution is

$$Q_k = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right)w,$$

where, L = lower boundary of the class containing the median or median class, cf = the cumulative frequency in the class preceding the class containing the median, f = the number of observations (frequency) in the class containing the median and w = the size of the class interval; that is, the width of the median class.

Example 3

Consider the following two sets of data:

A: $20, 70, 70, 70, 70, 70, 70, 70, 100$

B: $20, 30, 50, 80, 90, 100.$

a. Find the ranges of A and B.

b. Find the inter-quartile ranges of A and B.

Solution

a. The ranges of A and B are R = 100 – 20 = 80 and R = 100 – 20 = 80, respectively, from which you see that they have the same range. However, if you observe the two sets of data, you can see that B is more variable than A.

b.

For A:

$Q_1 = \frac{1(n+1)}{4} = \frac{9+1}{4} = \frac{10}{4}^{th}$ item=2.5^{th} .

Therefore, $Q_1 = 2^{nd}$ item+ 0.5(3^{nd} item-2^{nd} item) $= 70 + 0.5(70 - 70) = 70.$

$Q_3 = \frac{3(n+1)}{4} = \frac{3(9+1)}{4} = \frac{30}{4}^{th}$ item=7.5^{th}.

Therefore, $Q_3 = 7^{th}$ item+ 0.5(8^{th} item-7^{th} item) $= 70 + 0.5(70 - 70) = 70.$

Thus, $IQR = 70 - 70 = 0.$

For B:

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right)+\left(\frac{n}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{6}{4}\right)+\left(\frac{6}{4}+1\right)}{2}\right)^{th} = 2^{nd}\text{item}=30.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times6}{4}\right)+\left(\frac{3\times6}{4}+1\right)}{2}\right)^{th} \text{item} = 5^{th}\text{item} = 90.$$

Thus, $IQR = 90 - 30 = 60$.

From the above computation, you clearly see that B possesses higher variability than A. The greater the measure of variation, the higher the dispersion of the data set.

Limitation of Inter-Quartile Range

- It only depends on two values Q_1 and Q_3. It does not consider the variability of each item in the data set.
- It ignores 50% of the data (the top 25% above $Q_3$and the bottom 25% below Q_1. It only considers the middle 50% of the values between Q_1 and Q_3.

Exercise 3.2

1. Calculate the inter-quartile range of each of the following data sets:

   a. 25, 21, 22, 20, 19, 16 b. 14, 17, 18, 16, 10, 18, 19, 22.

2. Calculate the inter-quartile range for these data sets:

   a.

x	5	7	8	9	11
f	13	10	11	12	13

   b.

x	22	23	24	25
f	1	2	3	1

   c.

x	0-20	20-40	40-60	60-80	80-100
f	3	1	2	4	1

   d.

x	0-9	10-19	20-29	30-39
f	4	2	3	1

3.1.2 Mean Deviation and Quartile Deviation

Mean deviation for ungrouped data (1)

Activity 3.4

1. The following data set represents mathematics examination scores of fifteen students graded out of 15:

   10, 11, 12, 10, 9, 10, 12, 13, 8, 15, 14, 10, 9, 15, 10

   a. Find the mean of the data.

   b. Find the deviation of each value from the mean.

   c. Determine the mean of these deviations.

   d. Determine the mean of these deviations by considering the absolute values of each deviation.

   e. Compare the results you obtained in parts (c) and (d).

2. Find the mean, median and mode for the following data set.

x	10	11	12	13
f	3	1	2	4

3. Find the mean, median and mode for the following data describing the height of students in a certain class and given in continuous grouped data form as:

Height(cm)	150-154	155-159	160-164	165-169
f	3	1	2	4

Definition 3.4

Mean deviation of data is the sum of all deviations (in absolute value) of each item from the average value divided by the total number of items.

i. Mean deviation about the mean $MD(\bar{x})$

Steps to find or compute mean deviation from the mean:

Step 1: Find the mean of the data set.

Step 2: Find the deviation of each item from the mean regardless of sign (since mean deviation assumes absolute value).

Step 3: Find the sum of the deviations.

Step 4: Divide the sum by the total number of items in the data set.

That is,

$$MD(\bar{x}) = \frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + |x_3 - \bar{x}| + \ldots + |x_n - \bar{x}|}{n} = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

Example 1

Find the mean deviation about the mean of the following data.

50, 80, 80, 110, 110, 110, 140, 160.

Solution

Mean deviation about the mean

Step 1: Calculate the mean of the data set.

$$\bar{x} = \frac{50 + 80 + 80 + 110 + 110 + 110 + 140 + 160}{8} = \frac{840}{8} = 105$$

Step 2: Find the absolute deviation of each data item from the mean.

x	50	80	80	110	110	110	140	160
$\lvert x_i - \bar{x}\rvert$	55	25	25	5	5	5	35	55

Step 3: Find the sum of the deviations.

$$\sum |x_i - \bar{x}| = |50 - 105| + |80 - 105| + |80 - 105| + |110 - 105| + |110 - 105| + |110 - 105| + |140 - 105| + |160 - 105|$$

$$= 210.$$

Step 4: Divide the sum of the deviation by the total number of items in the data set.

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} = \frac{210}{8} = 26.25$$

Exercise 3.3

Calculate the mean deviation about the mean of the following data.

a. 15, 11, 12, 20, 19, 6, 11

b. 4, 7, 8, 9, 10, 10, 11, 12, 13, 17

Mean deviation for ungrouped data (2)

ii. Mean deviation about the median $MD(m_d)$

Here, we simply need to replace the role of the mean by the median and follow each step as above. This will give us the mean deviation about the median to be:

$$MD(m_d) = \frac{|x_1 - m_d| + |x_2 - m_d| + |x_3 - m_d| + \ldots + |x_n - m_d|}{n}$$

$$= \frac{\sum_{i=1}^{n} |x_i - m_d|}{n}$$

Example 2

Find the mean deviation about the median of the following data.

$50, 80, 80, 110, 110, 110, 140, 160.$

Solution

Step 1: Calculate the median of the data set.

$$m_d = \frac{\left(\frac{8}{2}\right)^{th} \text{item} + \left(\frac{8}{2}+1\right)^{th} \text{item}}{2} = \frac{110+110}{2} = 110$$

Step 2: Find the absolute deviation of each data item from the median.

x	50	80	80	110	110	110	140	160
$\lvert x_i - m_d \rvert$	60	30	30	0	0	0	30	50

Step 3: Find the sum of the deviations.

$$\sum_{i=1}^{n} |x_i - m_d| = |50-110| + |80-110| + |80-110| + |110-110| + |110 - 110| + |110-110| + |140-110| + |160-110| = 200.$$

Step 4: Divide the sum of the deviations by the total number of items in the data set.

$$MD(m_d) = \frac{\sum_{i=1}^{n} |x_i - m_d|}{8} = \frac{200}{8} = 25$$

Exercise 3.4

Calculate the mean deviation about the median of the following data.

a. 15, 11, 12, 20, 19, 6, 11

b. 4, 7, 8, 9, 10, 10, 11, 12, 13, 17

Mean deviation of ungrouped data (3)

iii. Mean deviation about the mode $MD(m_o)$

Again, proceed in a similar way as in the case of i and ii above. We get mean deviation about the mode to be:

$$MD(m_o) = \frac{|x_1 - m_o| + |x_2 - m_o| + |x_3 - m_o| + \ldots + |x_n - m_o|}{n}$$

$$= \frac{\sum_{i=1}^{n} |x_i - m_o|}{n}$$

Example 3

Find the mean deviation about the mode of the following data.

$50, 80, 80, 110, 110, 110, 140, 160.$

Solution

Mean deviation about the mode

Step 1: Identify the mode of the data set; mode = 110.

Step 2: Find the absolute deviation of each data item from the mode.

x	50	80	80	110	110	110	140	160
$\lvert x_i - m_o \rvert$	60	30	30	0	0	0	30	50

Step 3: Find the sum of the deviations.

$$\sum_{i=1}^{n} |x_i - m_o| = |50 - 110| + |80 - 110| + |80 - 110| + |110 - 110| + |110 - 110| + |110 - 110| + |140 - 110| + |160 - 110| = 200.$$

Step 4: Divide the sum of the deviations by the total number of items in the data set.

$$MD(m_o) = \frac{\sum_{i=1}^{n} |x_i - m_o|}{8} = 25$$

Exercise 3.5

Calculate the mean deviation about the mode of the following data.

a. 15, 11, 12, 20, 19, 6, 11

b. 4, 7, 8, 9, 10, 10, 11, 12, 13, 17

Mean deviation for discrete frequency distributions (1)

To calculate the mean deviation for a discrete frequency distribution about the mean, the median and the mode, you take similar steps as in the process for discrete data. If $x_1, x_2, \dots, x_n$ are values with corresponding frequencies $f_1, f_2, \dots, f_n$, then the mean deviation is computed as follows:

i. Mean deviation from the mean

To calculate the mean deviation from the mean, take the following steps:

Step 1: Find the mean of the data set.

Step 2: Find the deviation of each item from the mean regardless of sign (since mean deviation assumes absolute value).

Step 3: Multiply each deviation by its corresponding frequency.

Step 4: Find the sum of these deviations multiplied by their frequencies.

Step 5: Divide the sum by the sum of the frequencies in the data set.

$$MD(\bar{x}) = \frac{f_1|x_1 - \bar{x}| + f_2|x_2 - \bar{x}| + f_3|x_3 - \bar{x}| + \dots + f_n|x_n - \bar{x}|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i|x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

Example 4

Find the mean deviation of the following data about the mean.

x	f
6	2
10	6
12	11
20	13
25	8
26	2
30	4

Solution

Mean deviation of the data about the mean

Step 1: Calculating the mean:

$$\bar{x} = \frac{2\times\ 6 + 6\times\ 10 + 11\times\ 12 + 13\times 20 + 8\times\ 25 + 2\times\ 26 + 4\times 30}{2 + 6 + 11 + 13 + 8 + 2 + 4} = 18.2.$$

Step 2: Calculate the deviations from the mean:

x	f	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
6	2	12.2	24.4
10	6	8.2	49.2
12	11	6.2	68.2
20	13	1.8	23.4
25	8	6.8	54.4
26	2	7.8	15.6
30	4	11.8	47.2
Total	$\sum f = 46$		$\sum f\lvert x-\bar{x}\rvert = 282.4$

Step 3: Thus, mean deviation about the mean will be

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f_i\lvert x_i - \bar{x}\rvert}{\sum_{i=1}^{n} f_i} = \frac{282.4}{46} = 6.1.$$

Exercise 3.6

Calculate the mean deviation about the mean of the following data.

x	f
10	2
20	4
30	7
40	6
50	1

Mean deviation for discrete frequency distributions (2)

ii. Mean deviation about the median

Here, we simply need to replace the role of the mean by the median and follow each step as above. This will give us the mean deviation about the median to be:

$$MD(m_d) = \frac{f_1|x_1 - m_d| + f_2|x_2 - m_d| + f_3|x_3 - m_d| + \ldots + f_n|x_n - m_d|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i|x_i - m_d|}{\sum_{i=1}^{n} f_i}$$

Note that: Median for ungrouped data is:

a. $m_d = \left(\frac{(n+1)}{2}\right)^{th}$ item if n is odd

b. $m_d = \left(\frac{\left(\frac{n}{2}\right)+\left(\frac{n}{2}+1\right)}{2}\right)^{th}$ item if n is even.

Example 5

Find the mean deviation of the following data about the median.

x	f
6	2
10	6
12	11
20	13
25	8
26	2
30	4

Solution

Step 1: Calculating the median:

The median

$$m_d = \frac{\left(\frac{46}{2}\right)^{th} \text{item} + \left(\frac{46}{2}+1\right)^{th} \text{item}}{2} = \frac{20+20}{2} = 20$$

Step 2: Calculate the deviations from the median:

x	f	$\lvert x-m_d\rvert$	$f\lvert x-m_d\rvert$
6	2	14	28
10	6	10	60
12	11	8	88
20	13	0	0
25	8	5	40
26	2	6	12
30	4	10	40
Total	$\sum f = 46$		268

Step 3: Thus, the mean deviation about the median will be

$$MD(m_d) = \frac{\sum_{i=1}^{n} f_i|x_i - m_d|}{\sum_{i=1}^{n} f_i} = \frac{268}{46}$$

$$= 5.8.$$

Exercise 3.7

Calculate the mean deviation about the median of the following data.

x	f
10	2
20	4
30	7
40	6
50	1

Mean deviation for Discrete Frequency Distributions (3)

iii. Mean deviation about the mode

The steps that we need to follow here are also the same, but we shall use the mode instead of the mean or the median. Following the steps, we will get the mean deviation about the mode to be:

$$MD(m_o) = \frac{f_1|x_1 - m_o| + f_2|x_2 - m_o| + f_3|x_3 - m_o| + \ldots + f_n|x_n - m_o|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i|x_i - m_o|}{\sum_{i=1}^{n} f_i}$$

Example 6

Find the mean deviation of the following data set about the mode.

x	f
6	2
10	6
12	11
20	13
25	8
26	2
30	4

Solution

Step 1: Calculating the mode:

The mode $m_o = 20$.

Step 2: Calculate the deviations from the mode:

x	f	$\lvert x-m_o\rvert$	$f\lvert x-m_o\rvert$
6	2	14	28
10	6	10	60
12	11	8	88
20	13	0	0
25	8	5	40
26	2	6	12
30	4	10	40
Total	$\sum f = 46$		$\sum f\lvert x - m_o\rvert = 268$

Step 3: Thus, the mean deviation about the mode will be

$$MD(m_o) = \frac{\sum_{i=1}^{n} f_i |x_i - m_o|}{\sum_{i=1}^{n} f_i} = \frac{268}{46} = 5.8.$$

Exercise 3.8

Calculate the mean deviation about the mode of the following data.

x	f
10	2
20	4
30	7
40	6
50	1

Mean deviation for grouped frequency distributions (1)

a. Mean deviation of the data about the mean

For continuous grouped frequency distributions, mean deviation is calculated in the same way as the above except that each x_i is substituted by the midpoint (m_i) of each class.

This will give us the mean deviation about the mean to be:

$$MD(\bar{x}) = \frac{f_1|m_1 - \bar{x}| + f_2|m_2 - \bar{x}| + f_3|m_3 - \bar{x}| + \ldots + f_n|m_n - \bar{x}|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i |m_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

Example 7

Find the mean deviation about the mean for the following.

x	0-5	6-11	12-17	18-23	24-29
f	5	8	7	10	3

Solution

First, you have to find the mean of the distribution

x	f	m	fm	Cf
0-5	5	2.5	12.5	5
6-11	8	8.5	68	13
12-17	7	14.5	101.5	20
18-23	10	20.5	205	30
24-29	3	26.5	79.5	33
Total	$\sum f = 33$		$\sum fm = 466.5$	

Step 1: Calculating the mean.

$$\text{The mean} = \bar{x} = \frac{\sum fm}{\sum f} = \frac{466.5}{33} = 14.14$$

Step 2: Calculate the deviations from the mean:

x	f	m	$f\lvert m-\bar{x}\rvert$
0-5	5	2.5	58.20
6-11	8	8.5	45.12
12-17	7	14.5	2.52
18-23	10	20.5	63.6
24-29	3	26.5	37.08
Total	$\sum f = 33$		$\sum f\lvert m-\bar{x}\rvert = 206.52$

Step 3: Thus, mean deviation about the mean will be

$$\frac{\sum_{i=1}^{n} f_i\lvert m_i - \bar{x}\rvert}{\sum_{i=1}^{n} f_i} = \frac{206.52}{33} = 6.26.$$

Exercise 3.9

Calculate the mean deviation about the mean of the following data set.

x	0-9	10-19	20-29	30-39	40-49
f	4	3	6	5	2

Mean deviation for grouped frequency distributions (2)

b. Mean deviation of the data about the median

The mean deviation about the median is given by:

$$MD(m_d) = \frac{f_1|m_1 - m_d| + f_2|m_2 - m_d| + f_3|m_3 - m_d| + \ldots + f_n|m_n - m_d|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i|m_i - m_d|}{\sum_{i=1}^{n} f_i}$$

Note that the median of grouped data is given by

$$m_d = L + \left(\frac{\frac{n}{2} - cf}{f}\right) w$$

where, L = lower boundary of the class containing the median or median class, cf = the cumulative frequency in the class preceding the class containing the median, f = the number of observations (frequency) in the class containing the median and w = the size of the class interval that is, the width of the median class.

Example 8

Find the mean deviation about the median for the following.

x	0-5	6-11	12-17	18-23	24-29
f	5	8	7	10	3

Solution

Step 1: Calculating the median:

$$m_d = L + \frac{(\frac{n}{2} - cf)}{f} = 11.5 + 5\left(\frac{16.5 - 13}{7}\right) = 14$$

Step 2: Calculate the deviations from the median:

x	f	m	Cf	$f\|m\text{-}m_d\|$
0-5	5	2.5	5	57.5
6-11	8	8.5	13	44
12-17	7	14.5	20	3.5
18-23	10	20.5	30	65
24-29	3	26.5	33	37.5
Total	$\sum f = 33$			$\sum f\|m - m_d\| = 207.5$

Step 3: Thus, mean deviation about the median will be

$$\frac{\sum_{i=1}^{n} f_i|m_i - m_d|}{\sum_{i=1}^{n} f_i} = \frac{207.5}{33} = 6.29.$$

Exercise 3.10

Calculate the mean deviation about the median of the following data.

x	0-9	10-19	20-29	30-39	40-49
f	4	3	6	5	2

Mean deviation for grouped frequency distributions (3)

c. Mean deviation of the data about the mode

The mean deviation about the mode is:

$$MD(m_o) = \frac{f_1|m_1 - m_o| + f_2|m_2 - m_o| + f_3|m_3 - m_o| + \ldots + f_n|m_n - m_o|}{f_1 + f_2 + f_3 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i|m_i - m_o|}{\sum_{i=1}^{n} f_i}$$

Note that the mode of grouped data given by

$$m_o = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) w$$

where, L = lower class boundary of the modal class, Δ_1 = the difference between the frequency of the modal class (f_{m_o}) and the frequency of the preceding class (pre-modal class) = $f_{m_o} - f_1$

Δ_2 = the difference between the frequency of the modal class and the frequency of the subsequent class (next class) = $f_{m_o} - f_2$ and w= size of the class interval.

Example 9

Find the mean deviation about the mode for the following.

x	0-5	6-11	12-17	18-23	24-29
f	5	8	7	10	3

Solution

Step 1: Calculating the mode:

The mode $m_o = L + (\frac{\triangle_1}{\triangle_1 + \triangle_2})w = 17.5 + \frac{3}{3+7} = 19.3.$

Step 2: Calculate the deviations from the mode.

X	f	m	$f\|m-m_o\|$
0-5	5	2.5	82.5
6-11	8	8.5	84
12-17	7	14.5	31.5
18-23	10	20.5	15
24-29	3	26.5	22.5
Total	$\sum f = 33$		$\sum f\|m - m_o\| = 235.5$

Step 3: Thus, mean deviation about the mode will be

$$\frac{\sum_{i=1}^{n} f_i|m_i - m_o|}{\sum_{i=1}^{n} f_i} = \frac{235.5}{33} = 7.1.$$

Exercise 3.11

Calculate the mean deviation about the mode of the following data.

x	0-9	10-19	20-29	30-39	40-49
f	4	3	6	5	2

Advantage and Limitation of Mean Deviation

From the above discussions on mean deviation, it can be noticed that mean deviation can be useful for applications. If our average is "Arithmetic mean", you take the deviation about the mean. Similarly, if our average is "Median", then you take the deviation about the median and if our average is the "Mode", you take mean deviation about the mode. To decide which one of the mean deviations to use in a given situation, consider the following points:

i. If the degree of variability in a set of data is not very high, use of the mean deviation about the mean since it is comparatively the best for interpretation.

ii. Whenever there is an extreme value that can affect the mean, however, the mean deviation about the median is preferable.

Though mean deviation has some advantages, it is not commonly used for interpretation. Rather, it is the standard deviation that is commonly used and which tends to be the best measure of variation.

Advantages of mean deviation

- Compared to range, mean deviation has some advantages. Range considers only two values, whereas mean deviation takes each value into consideration.

Limitations of mean deviation

- It ignores signs of deviation by taking the absolute value of a deviation which violates the rules of algebra.

Exercise 3.12

1. Calculate the mean deviation about the mean, median and mode of each of the following data sets:

   a. 12, 9, 15, 12, 7, 10, 12.
   b. 4, 7, 8, 9, 10, 10, 11, 12, 13, 17

2. Calculate the mean deviation about the mean, median and mode for the following data sets:

   a.

x	12	13	14	15
f	10	12	9	11

   b.

x	0-4	5-9	10-14	15-19	20-24
f	3	1	2	4	1

Quartile deviation of ungrouped data

Activity 3.5

A certain shop has registered the following data on daily sales (in 100 Birr) for one week:

$300, 450, 540, 600, 250, 350, 420$

1. Calculate the range and inter-quartile range of the data.
2. Compare the values you obtained in question 1.

Definition 3.5

Quartile Deviation (QD) is half of the difference between the upper and lower quartiles, that is, $QD = \frac{Q_3 - Q_1}{2}$.

Note

Steps to calculate quartile deviation from ungrouped data:

Step 1: Arrange the data in ascending order.

Step 2: Assign serial numbers to each item. The first serial number is assigned to the lowest test score, while the last serial number is assigned to the highest item.

Step 3: Determine the first quartile Q_1 and third quartile Q_3.

Step 4: Use the obtained values in locating the serial numbers of the item that falls under Q_1 and Q_3.

Step 5: Subtract Q_1 from Q_3 and divide the difference by 2.

Example 10

Find the quartile deviation of the following ungrouped data.

$96, 70, 100, 89, 78, 56, 45, 78, 68, 42, 66, 89, 90, 54, 44, 67, 87, 97, 98$

Solution

Step 1: First arrange the data in ascending order:

$42, 44, 45, 54, 56, 66, 67, 68, 70, 78, 78, 87, 89, 89, 90, 96, 97, 98, 100.$

Step 2: Assign serial numbers to each score

x	42	44	45	54	56	66	67	68	70	78	87	87	89	89	90	96	97	98	100
S No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Step 3: Determine the first quartile (Q_1) and third quartile (Q_3).

$$Q_1 = \frac{1(n+1)}{4} = \frac{19+1}{4} = \frac{20^{th}}{4} \text{ item} = 5^{th} \text{item and}$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(19+1)}{4} = \frac{60^{th}}{4} \text{ item} = 15^{th} \text{item.}$$

Step 4: Use the obtained values in locating the serial numbers of the score that falls under (Q_1) and (Q_3); that is,

5^{th}item $= 56$ and 15^{th}item $= 90$.

Step 5: Subtract (Q_1) from (Q_3) and divide the difference by 2.

$$QD = \frac{Q_3 - Q_1}{2} = \frac{90-56}{2} = \frac{34}{2} = 17.$$

Exercise 3.13

Calculate the quartile deviation of each of the following data sets.

a. 105, 101, 102, 109, 60, 100, 120

b. 40, 70, 60, 90, 110, 100, 111, 120, 130, 170

Quartile deviation of grouped data

The formula for i^{th} quartile is $Q_i = L + \left(\dfrac{\frac{i(n)}{4} - cf}{f}\right) \times h,\ i = 1, 2, 3,$

where

L: The lower limit of the i^{th} quartile class

$n = \sum f$: Total number of observations

f : Frequency of the i^{th} quartile class

cf : Cumulative frequency of the class previous to i^{th} quartile class

h : The class width

Example 11

The following table gives the amount of time (in minutes) spent on the internet each evening by a group of 56 students. Calculate the inter-quartile range for the following frequency distribution.

Time spent on internet (x)	10-12	13-15	16-18	19-21	22-24
Number of students (f)	3	12	15	24	2

Solution

Class interval	*Class Boundary*	*f*	*cf*
10-12	9.5-12.5	3	3
13-15	12.5-15.5	12	15
16-18	15.5-18.5	15	30
19-21	18.5-21.5	24	54
22-24	21.5-24.5	2	56
Total		56	

In order to evaluate the Quartile Deviation of grouped data, first we find

$$Q_i = \left(\frac{i(n)}{4}\right)^{th} \text{ value}, \quad i = 1, 2, 3,$$

where n is the total number of observations.

The first Quartile Q_1 is calculated as

$$Q_1 = \left(\frac{i(n)}{4}\right)^{th} \text{ value} = \left(\frac{1(56)}{4}\right)^{th} \text{ value} = (14)^{th} \text{ value.}$$

The cumulative frequency just greater than or equals to 14 is 15. The corresponding class, 12.5-15.5 is the first quartile class. Thus,

$L = 12.5,$ the lower limit of the 1^{st} quartile class

$n = 56,$ total number of observations

$f = 12,$ frequency of the 1^{st} quartile class

$cf = 3,$ cumulative frequency of the class previous to the 1^{st} quartile class

$h = 3,$ the class width

The first quartile Q_1 can be computed as follows:

$$Q_1 = L + \left(\frac{\frac{i(n)}{4} - cf}{f} \right) \times h$$

$$= 12.5 + \left(\frac{\frac{1(56)}{4} - 3}{12} \right) \times 3 = 12.5 + \left(\frac{14 - 3}{12} \right) \times 3$$

$$= 12.5 + 2.75$$

$$= 15.25 \text{ minutes}$$

Thus, 25% of the students spent less than or equal to 15.25 minutes on the internet.

The third Quartile Q_3 is calculated as

$$Q_3 = \left(\frac{i(n)}{4} \right)^{th} \text{value} = \left(\frac{3(56)}{4} \right)^{th} \text{value} = (42)^{th} \text{value.}$$

The cumulative frequency just greater than or equals to 42 is 54. The corresponding class is 18.5-21.5 is the third quartile class. Thus,

$L = 18.5,$ the lower limit of the 3^{rd} quartile class

$n = 56,$ total number of observations

$f = 24,$ frequency of the 3^{rd} quartile class

$cf = 30,$ cumulative frequency of the class previous to the 3^{rd} quartile class

$h = 3,$ the class width

The third quartile Q_3 can be computed as follows:

$$Q_3 = L + \left(\frac{\frac{i(n)}{4} - cf}{f} \right) \times h$$

$$= 18.5 + \left(\frac{\frac{3(56)}{4} - 30}{24} \right) \times 3 = 18.5 + \left(\frac{42 - 30}{24} \right) \times 3$$

$$= 18.5 + 1.5$$

$$= 20 \text{ minutes}$$

Thus, 75% of the students spent less than or equal to 20 minutes on the internet.

Therefore, the Quartile Deviation (QD) is calculated as

$$QD = \frac{Q_3 - Q_1}{2} = \frac{20 - 15.25}{2} = 2.375 \text{ minutes.}$$

Exercise 3.14

Calculate the quartile deviation for the following data sets.

a.

x	110	130	140	150
f	2	3	2	1

b.

x	0-4	5-9	10-14	15-19	20-24
f	3	1	2	4	1

3.1.3 Variance and Standard Deviation

Variance for ungrouped data

Activity 3.6

The following are the average daily temperatures (in °C) for one week in a certain city: **5, 16, 16, 10, 17, 20, 14**

1. Calculate the range, inter-quartile range and mean deviation about the mean of the data.
2. Compare the results you obtained in (1).

In the previous section, we discussed measures of absolute dispersions or variations. In this section, we will discuss about dispersions or variation of items/values in a data set in terms of the average/mean of the data. That is, in order to measure the extent by which the distributions of the data values are dispersed or spread from the mean, such variability is defined in definition 3.6.

Definition 3.6

The variance of a data set is the average of the squared difference of each value from the mean of the data set.

Raw data is a list of values which are given in raw form.

For Example: 6, 5, 9, 13, 11, 8, 9, 9

If $x_1, x_2, \dots, x_n$ are n observed values, then variance σ^2 for the sample data is given by

$$\sigma^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots . + (x_n - \bar{x})^2}{n} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

where, n = number of values and $\bar{x}$ = mean.

Note

Steps to find variance for ungrouped data:

Step 1: Find the mean of the data.

Step 2: Find the deviation of each value from the mean and square it.

Step 3: Add the squared deviations.

Step 4: Divide the sum obtained in step 3 by n.

Example 1

Find the variance of the following data.

$$16, 12, 8, 18, 5, 9, 24, 20.$$

Solution

Calculate the mean. The mean $(\overline{x})$ is:

$$\bar{x} = \frac{16 + 12 + 8 + 18 + 5 + 9 + 24 + 20}{8} = 14$$

Calculate the deviation of each value from the mean and square it.

x	$x-\overline{x}$	$(x-\overline{x})^2$
16	2	4
12	-2	4
8	-6	36
18	4	16
5	-9	81
9	-5	25
24	10	100
20	6	36
Total		$\sum(x_i - \bar{x})^2 = 302$

Thus, $\sigma^2 = \frac{\sum(x-\bar{x})^2}{n} = \frac{302}{8} = 37.75.$

Exercise 3.15

Find the variance of the following data set:

5, 10, 4, 7, 9

Variance for ungrouped frequency distribution

Ungrouped frequency distribution is a table that describes values a certain data with their respective frequencies and listed in ascending or descending order. For example, given value and frequency like the following table.

Value(v)	x_1	x_2	x_3	…	x_n
Frequency(f)	f_1	f_2	f_3	…	f_n

If $x_1, x_2, \ldots, x_n$ are values with corresponding frequencies $f_1, f_2, \ldots, f_n$, the variance is given by

$$\sigma^2 = \frac{f_1(x_1-\bar{x})^2 + f_2(x_2-\bar{x})^2 + f_3(x_3-\bar{x})^2 + \cdots . + f_n(x_n-\bar{x})^2}{f_1+f_2+f_3+\cdots+f_n}$$

$$= \frac{\sum_{i=1}^{n} f_i(x_i-\bar{x})^2}{\sum_{i=1}^{n} f_i}$$

where, n = number of values and $\bar{x}$ = mean.

Note

Steps to compute/find variance from ungrouped frequency:

Step 1: Find the mean of the distribution.

Step 2: Find the deviation of each value from the mean and square it.

Step 3: Multiply the squared deviations by their corresponding frequencies and add them.

Step 4: Divide the sum obtained in step 3 by $\sum f_i$.

Example 2

Find the variance of the following data set.

x	2	3	4	6
f	5	4	2	1

Solution

Calculate the mean of the distribution.

x	f	$x-\bar{x}$	$f(x-\bar{x})^2$
2	5	-1	5
3	4	0	0
4	2	1	2
6	1	3	9
Total	$\sum f_i = 12$		$\sum f(x_i-\bar{x})^2 = 16$

The mean

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{36}{12} = 3$$

Calculate the deviation of each value from the mean and square it.

The required variance

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i} = \frac{16}{12} = 1.33.$$

Exercise 3.16

Find the variance of the following data set:

x	10	20	30	40	50
f	4	6	5	3	2

Variance for grouped data

Before we look at finding variance for grouped data, let us state the basic steps that are similar to variance for ungrouped data as follows:

Note

Steps to find variance from a grouped frequency distribution:

Step 1: Find the class mark for each class.

Step 2: Find the mean of the grouped data.

Step 3: Find the deviation of each class mark from the mean and square it.

Step 4: Find the sum of the squared deviations multiplied by their respective frequencies.

Step 5: Divide the sum obtained in step 4 by $\sum f$

Therefore, based on the above steps, the variance for grouped data is obtained by using the formula:

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i(m_i - \bar{x})^2}{\sum_{i=1}^{n} f_i}$$

where m_i is the midpoint of each class or class mark.

Example 3

Find the variance of the following data set.

x	0-4	4-8	8-12	12-16
f	4	8	2	1

Solution

Calculate the mean of the distribution.

$$\bar{x} = \frac{\sum f_i m_i}{\sum f_i} = \frac{90}{15} = 6$$

Calculate the deviation of each value from the mean and square it.

x	f	m_i	$f(m-\bar{x})^2$
0-4	4	2	64
4-8	8	6	0
8-12	2	10	32
12-16	1	14	64
Total	$\sum f = 15$		$\sum f(m_i-\bar{x})^2 = 160$

The required variance is, therefore,

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i(m_i-\bar{x})^2}{\sum_{i=1}^{n} f_i} = \frac{160}{15} = 10.67.$$

Exercise 3.17

Find the variance of the following data.

x	0-10	10-20	20-30	30-40	40-50
f	2	8	5	4	1

Standard deviation

Activity 3.7

The following data represent the daily sales (in 1000 Birr) of certain supermarket for ten days. **30, 45, 54, 60, 25, 35, 42, 80, 70, 40**

a. Compute the different measures of dispersion (mean, range, inter-quartile range, quartile deviation).

b. Discuss similarities and differences between the different measures of dispersion.

From the previous discussion, you know that mean deviation considers all the data values. However, the mean deviation assumes only the absolute deviations of each data value from the central value (mean, median or mode). Hence, it misses algebraic considerations. To overcome the limitation of mean deviation, a better measure of variation which is known as standard deviation can be used.

The standard deviation is the most commonly used measure of dispersion. The value of the standard deviation tells us how closely the values of a data set are clustered around the mean. In general, a lower value of the standard deviation for a data set indicates that the values of the data set are spread over a relatively small range around the mean. On the other hand, a large value of the standard deviation for a data set indicates that the values of that data set are spread over a relatively large range around the mean.

Definition 3.7

The standard deviation of a data set is the positive square root of a variance. The standard deviation is represented by the Greek σ (sigma).

If $x_1,\ x_2,\ \ldots, x_n$ are n observed values, then standard deviation for the ungrouped data is given by $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$ where, n = number of values and $\bar{x}$ = mean.

The standard deviation for grouped data is given by $\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i(m_i-\bar{x})^2}{n}}$, where m_i is the midpoint of each class or class mark and f_i is the frequency of group i.

Example 4

Find the standard deviation of the data in Examples 1, 2 and 3 above.

Solution

Standard deviation of the data in Example 1 is $\sigma = \sqrt{37.75} = 6.14$, standard deviation of the data in Example 2 is $\sigma = \sqrt{1.33} = 1.15$ and standard deviation of the data in Example 3 is $\sigma = \sqrt{10.67} = 3.27$.

Advantages and limitations of standard deviation

Advantages of standard deviation

- It is rigidly defined.
- It is based on all observations.

Limitations of standard deviation

- The process of squaring deviations and then taking the square root of their mean is complicated.
- It attaches great weight to extreme values since the squared deviations are used.

Exercise 3.18

1. Determine the variance and standard deviation of the following data set:

$$6, 7, 4, 11, 10, 11, 12, 13, 17$$

2. Calculate the variance and standard deviation for these data sets:

a.

x	1	2	3	5	6	7
f	2	3	2	1	2	2

b.

x	10-19	20-29	30-39	40-49
f	2	3	1	2

3.2 Interpretation of Relative Dispersions

Activity 3.8

1. Why do we study relative dispersion?
2. Two basketball teams scored the following points in ten different games:

   Team A: 30, 91, 0, 64, 42, 80, 30, 5, 117, 71

   Team B: 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

   a. Calculate the mean of each team's scores.

   b. Using the results obtained in part (a), compare and discuss the performance of the two teams.

In statistics, measures of dispersions help us to interpret the variability of the values in a data set; that is, they enable us to measure how homogeneous or heterogeneous the values in a data set are. In simple terms, they show how squeezed or scattered the values in a data set are.

3.2.1 Coefficient of Range

Activity 3.9

Consider the following two sets of data:

Data A: $2, 7, 7, 7, 7, 7, 7, 10$

Data B: $2, 3, 5, 8, 9, 10$

1. Calculate the range of each data set.
2. Based on the value you obtained in question 1, what is your interpretation of the two data?

Coefficient of range is a relative measure based on the value of range and defined as follows:

Definition 3.8

Coefficient of Range $CR = \frac{L-S}{L+S}$, where L = largest value and S = smallest value.

Example 1

Let us take two sets of observations of marks of students. Set A contains the marks of seven students in Physics out of 25 and set B contains marks of the same number of students in Mathematics out of 50 as shown below:

Physics: 10, 15, 18, 20, 20, 21, 21

Mathematics: 20, 25, 26, 27, 30, 35, 35

a. Find the ranges of the sets A and B.

b. Calculate the coefficients of ranges of the sets A and B.

Solution

Physics: 10, 15, 18, 20, 20, 21, 21

Mathematics: 20, 25, 26, 27, 30, 35, 35

The range and coefficient of range are calculated as:

Subjects	Range	CR
Physics	21-10=11	$\frac{21-10}{21+10}=0.35$
Mathematics	35-20=15	$\frac{35-20}{35+20}=0.27$

From the above computation, we can observe that the range is 11 in Physics and that the range is 15 in Mathematics. This is based on absolute measures of dispersion but not relative measures of dispersion. However, the reality is that the two subjects cannot be compared directly as their base is not the same. When we convert these two values into coefficients of range, we see that the coefficient of range for Physics is greater than that of Mathematics. Thus, there is greater dispersion or variation in Physics than Mathematics. This implies that the marks of students in Mathematics are more stable than their marks in Physics. We will learn how to make interpretations like this using relative measures of dispersion.

Example 2

a. Find the coefficient of range of the following data set:

$22, 28, 31, 23, 24.$

b. From the following data, calculate the coefficient of range.

x	5	15	25	35	45	55
f	10	10	30	50	40	30

c. From the following data, calculate coefficient of range.

Mark	0-10	10-20	20-30	30-40	40-50	50-60
No. students	1	5	3	4	5	2

Solution

a. $L = 31$ and $S = 22$, then $CR = \frac{L-S}{L+S} = \frac{31-22}{31+22} = \frac{9}{53} = 0.17$.

b. $L = 55$ and $S = 5$, then $CR = \frac{L-S}{L+S} = \frac{55-5}{55+5} = \frac{50}{60} = 0.83.$

c. Upper limit of highest class $L = 60$ and lower limit of lower class $S = 0$, then

$$CR = \frac{L-S}{L+S} = \frac{60-0}{60+0} = 1.$$

Exercise 3.19

1. Consider the following data which shows the hourly wages earned (in Birr) by ten workers

$$42, 17, 83, 59, 72, 76, 64, 45, 40, 32.$$

Find the CR of the data.

2. Find the CR of the following distribution.

x	10	15	17	18	20
f	3	4	9	3	6

3. The large L and small S values of gross incomes of two companies (in Birr) are given below:

Company	S	L
A	1000	43200
B	1200	36000

a. Calculate the CR of each company.

b. Which company has the more variable income?

3.2.2 Coefficient of Quartile Deviation (CQD)

Coefficient of quartile deviation (1)

Activity 3.10

Consider the following two data sets.

A: $150, 180, 190, 30, 20, 70, 100$

B: $130, 160, 150, 150, 140, 160, 170$

1. Find the first and the third quartiles for each data set.
2. Determine the range and inter-quartile range for each data set.
3. Compare which set of data has higher variation.

The coefficient of quartile deviation is a measure of relative dispersion and is based on the upper quartile Q_3 and the lower quartile Q_1 is defined as follows:

Definition 3.9

Coefficient of Quartile Deviation: $CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1}$

Example 1

For the data given in Example 1 in the previous section, calculate the coefficient of quartile deviation of both subjects.

Solution

Physics: $10, 15, 18, 20, 20, 21, 21$

Mathematics:$20, 25, 26, 27, 30, 35, 35$

For mathematics:

$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = \left(\frac{8}{4}\right)^{th}$ item=2^{nd}item $= 25$ and

$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = \left(\frac{24}{4}\right)^{th}$ item=6^{th} item $= 35$

Thus, coefficient of quartile deviation of mathematics is

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{35 - 25}{35 + 25} = \frac{10}{60} = 0.17.$$

For physics:

$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = \left(\frac{8}{4}\right)^{th}$ item=2^{nd}item $= 15$ and

$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = \left(\frac{24}{4}\right)^{th}$ item=6^{th} item $= 21$

Thus, coefficient of quartile deviation of physics is

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{21 - 15}{21 + 15} = \frac{6}{36} = 0.17.$$

We see from the above computation that coefficient of quartile deviation for both Physics and Mathematics is 0.17 for both subjects. Therefore, the inference is that the marks or scores of students in both subjects indicate the same median performance.

Example 2

The number of complaints received by the manager of Geda supermarket was recorded for each of the last 10 working days as it is given below:

$21, 15, 18, 10, 5, 17, 21, 19, 28, 25.$

Find coefficient of quartile deviation of the data.

Solution

Sorted data: $5, 10, 15, 17, 18, 19, 21, 21, 25, 28.$

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right) + \left(\frac{n}{4} + 1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{10}{4}\right) + \left(\frac{10}{4} + 1\right)}{2}\right)^{th} = 3^{rd} \text{ item} = 15.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4} + 1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times10}{4}\right) + \left(\frac{3\times10}{4} + 1\right)}{2}\right)^{th} \text{ item}$$

$$= 8^{th} \text{ item} = 21.$$

Thus, coefficient of quartile deviation is: $CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{21-15}{21+15} = \frac{6}{36} = 0.17.$

Exercise 3.20

1. The marks obtained, in English, are by 9 students given below. Calculate the coefficient of quartile deviation for data.

   $45, 32, 37, 46, 39, 36, 41, 48, 36.$

2. Calculate the coefficient of quartile deviation for data set: 2, 4, 6, 8, 10, 5, 6, 9, 4, 6.

Coefficient of quartile deviation (2)

Example 3

From the following data, calculate the coefficient of quartile deviation.

Mark	0-10	10-20	20-30	30-40	40-50	50-60
No. students	10	20	30	50	40	30

Solution

Calculation of cumulative frequency

Mark	No. students	*cf*
0-10	10	10
10-20	20	30
20-30	30	60
30-40	50	110
40-50	40	150
50-60	30	180

$$Q_1 = \left(\frac{n}{4}\right)^{th} = \left(\frac{180}{4}\right)^{th} = 45^{th}\text{item.}$$

Thus, Q_1 lies in the class 20-30 and hence,

$$Q_1 = L + \frac{(\frac{n}{4} - cf)w}{f} = 20 + \frac{(\frac{180}{4} - 30)10}{30} = 20 + \frac{150}{30} = 25.$$

$$Q_3 = \left(\frac{3n}{4}\right)^{th} = \left(\frac{540}{4}\right)^{th} = 135^{th}\text{item}$$

Thus, Q_3 lies in the class 40-50 and hence

$$Q_3 = L + \frac{(\frac{3n}{4} - cf)w}{f} = 40 + \frac{(\frac{3(180)}{4} - 110)10}{40} = 40 + \frac{250}{40} = 46.25.$$

Therefore, coefficient of quartile deviation of the data is

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{46.25 - 25}{46.25 + 25} = \frac{21.25}{71.25} = 0.3.$$

Exercise 3.21

1. Calculate the coefficient of quartile deviation for the continuous grouped data sets given below:

Class boundaries	*f*
29.5-39.5	8
39.5-49.5	87
49.5-59.5	190
59.5-69.5	304
69.5-79.5	211
79.5-89.5	85
89.5-99.5	20

2. The Q_1 and Q_3 of gross incomes of two companies are given below:

Company	Q_1	Q_3
A	5000	54300
B	1000	45320

a. Calculate the *CQD* of each company.

b. Which company has more variable income?

3.2.3 Coefficient of Mean Deviation (CMD)

Activity 3.11

Consider the following data which shows the amount of sugar, in kilograms, sold by a small shop over for ten days:

$$4, 5, 6, 4, 6, 3, 7, 6, 3, 5$$

1. Determine the value of the mean deviation of the data about the mean, median and mode.
2. Compare the values you obtained in question 1.

The coefficient of quartile deviation is a measure of relative dispersion that is based on the mean deviation. It may be taken from the mean, median or mode and is defined as follows:

Definition 3.10

The coefficient of Quartile Deviation from the mean is $CMD = \frac{MD}{\bar{x}}$, where *MD* = mean deviation from the mean and $\bar{x}$ = mean.

Definition 3.11

The coefficient of Quartile Deviation from the median is $CMD = \frac{MD}{m_d}$, where *MD* = mean deviation from the median and m_d = median.

Definition 3.12

The coefficient of Quartile Deviation from the mode is $CMD = \frac{MD}{m_0}$, where $MD =$ mean deviation from the mode and $m_o =$ mode.

Example

Find the coefficient of mean deviation of the following data set about the mean.

x	0	1	2	3	4
f	1	9	7	3	4

Solution

First we calculate mean

$$\bar{x} = \frac{0 \times 1 + 1 \times 9 + 2 \times 7 + 3 \times 3 + 4 \times 4}{1 + 9 + 7 + 3 + 4} = \frac{48}{24} = 2.$$

x	f	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
0	1	2	2
1	9	1	9
2	7	0	0
3	3	1	3
4	4	2	8
Total	$\sum f = 24$		$\sum f\lvert x - \bar{x}\rvert = 22$

Thus, $MD = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{22}{24} = 0.92.$

Therefore, $CMD = \frac{MD}{\bar{x}} = \frac{0.92}{2} = 0.46.$

Exercise 3.22

1. Calculate the coefficient of quartile deviation from the mean of the following set of examination marks: $45, 32, 37, 46, 39, 36, 41, 48, 36.$
2. Find the coefficient of quartile deviation from the mode of the continuous grouped data given below:

Income	35-39	40-44	45-49	50-54	55-59	60-64	65-69
f	13	15	17	28	12	10	5

3. The median and mean deviation from the median of gross incomes of two companies are given below:

Company	m_d	$MD(m_d)$
A	50000	4430
B	20000	2200

a. Calculate the CMD of each company from the median.

b. Which company has more consistent income?

3.2.4 Coefficient of Variation (CV)

Activity 3.12

Find the variance σ^2 and the standard deviation σ for the following data set:

$$50, 30, 80, 110, 130.$$

The coefficient of variation is a measure of relative dispersion that is based on standard deviation. It is used to compare the variations or the performance of two sets of data. A large value indicates that there is greater variability while a small value indicates less variability.

Definition 3.13

The coefficient of Variation is a unit-less relative measure that is used to measure the degree of consistency given as a ratio of the standard deviation to the mean, that is, $CV = \frac{\sigma}{\bar{x}} \times 100$

Example 1

Calculate the variance, standard deviation and coefficient of variation from the following marks obtained by nine students.

$$45, 32, 37, 46, 39, 36, 41, 48, 36.$$

Solution

First we compute mean of the data.

$$\bar{x} = \frac{45+32+37+46+39+36+41+48+36}{9} = 40$$

Calculate the deviation for each value from the mean and square it.

x	$x-\bar{x}$	$(x-\bar{x})^2$
45	5	25
32	-8	64
37	-3	9
46	6	36
39	-1	1
36	-4	16
41	1	1
48	8	64
36	-4	16
Total		$\sum(x-\bar{x})^2 = 232$

Thus, the variance is $\sigma^2 = \frac{\sum(x-\bar{x})^2}{n} = \frac{232}{9} = 25.78.$

and the standard deviation $\sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{9}} = \sqrt{25.78} = 5.08.$

Therefore, $CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{5.08}{40} \times 100 = 12.7.$

Example 2

Calculate the coefficient of variation from the following mass (in grams) of 60 mangos:

Wight	65-84	85-104	105-124	125-144	145-164	165-184	185-204
f	9	10	17	10	5	4	5

Solution

x	m	f	$f(m-\bar{x})^2$
65-84	74.5	9	20736
85-104	94.5	10	7840
105-124	114.5	17	1088
125-144	134.5	10	1440
145-164	154.5	5	5120
165-184	174.5	4	10816
185-204	194.5	5	25920
Total		$\sum f = 60$	$\sum f(m-\bar{x})^2 = 72960$

First compute the mean of the data.

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{7350}{60} = 122.5.$$

Calculate the deviation for each mid-point from the mean and square it.

Calculate the variance: $\sigma^2 = \frac{\sum f(m-\bar{x})^2}{\sum f} = \frac{72960}{60} = 1216.$

Thus, $\sigma = \sqrt{\frac{\sum f(m-\bar{x})^2}{60}} = \sqrt{1216} = 34.87.$

Therefore, $CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{34.87}{122.5} \times 100 = 28.46.$

Exercise 3.23

1. Find the CV of the following distribution.

x	2	5	7	8	10
f	3	4	9	3	6

2. Consider the following data set which show the amount of milk sold, in liters, by two farmers over ten days.

   Farmer A: 42, 17, 83, 59, 72, 76, 64, 45, 40 ,32

   Farmer B: 28, 70, 31, 0, 59, 108 ,82, 14, 3, 95

   a. Calculate the CV for each farmer.

   b. Which farmer consistently sold more milk?

3. The mean and standard deviation of the gross incomes of two companies are given below:

Company	$\overline{x}$	σ
A	12000	2400
B	20000	4400

a. Calculate the CV of each company.

b. Which company has more variable income?

3.3 Use of Frequency Curves

The shape of a frequency curve describes the distribution of a data set. Such a description is made possible after the frequency curve of a frequency distribution is drawn. In this section, you will see how the measures of central tendency (mean, mode and median) determine the skewness of a distribution.

3.3.1 Mean, Median and Mode on the Frequency Curve

Activity 3.13

Consider the following data:

Data A: $20, 30, 40, 50, 50, 60, 50, 70, 80$

Data B: $20, 30, 10, 40, 80, 50, 80, 60, 80$

a. Calculate and compare the mean, median and mode for each data set.

b. Construct frequency curves for each data set and discuss your observations.

A measure of central tendency or a measure of dispersion alone does not tell us whether or not the distribution is symmetrical. It is the relationship between the mean, median and mode that tells us whether the distribution is symmetrical or skewed.

Example

Consider the following frequency distribution:

x	20-21	22-23	24-25	26-27	28-29
f	4	12	24	12	4

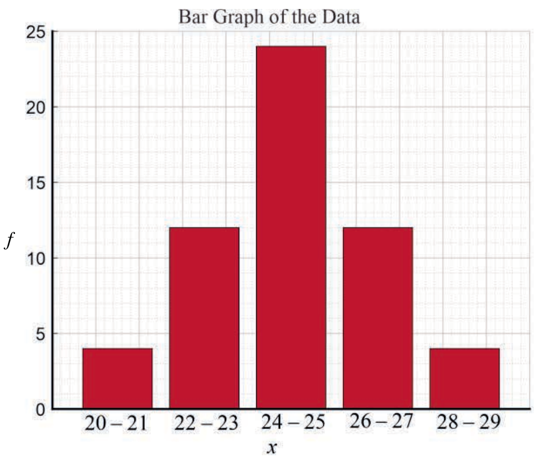
a. Draw the bar graph and frequency curve.

b. Calculate the mean, median and mode.

c. Describe the relationship between the mean, median and mode, and the skewness of the distribution.

Solution

a. The bar graph and frequency curve of this frequency distribution are as follows.



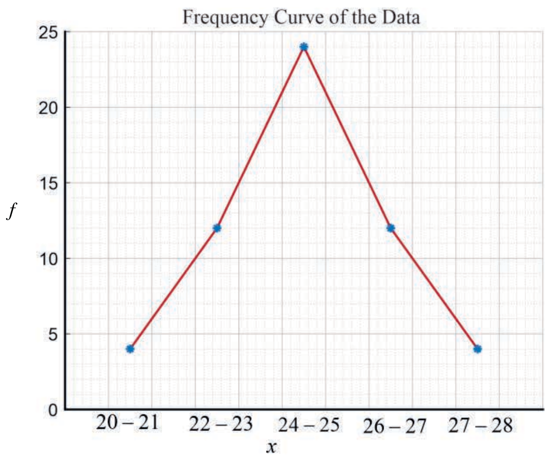


Figure 3.1

b. Mean = median = mode = 24.5.

c. From (a) and (b), you see that whenever mean = median = mode, then the distribution is perfectly symmetrical.

Investigate what happens to the skewness of a distribution, if mean > median > mode.

From the discussions outlined above, you can make the following generalizations:

a. If the mean is smallest in value, and the median is larger than the mean but smaller than the mode, then the distribution is negatively skewed. That is, if

Mean < Median < Mode then the distribution is negatively skewed (skewed to the left).

b. For a unimodal distribution in which the values of mean, median and mode coincide (that is, Mean = Median = Mode), the distribution is said to be perfectly symmetrical.

c. If the mean is the largest in value, and the median is larger than the mode but smaller than the mean, then the distribution is positively skewed. That is, if Mean> Median > Mode, then the distribution is positively skewed (skewed to the right).

Observe the following figure for a, b and c above.

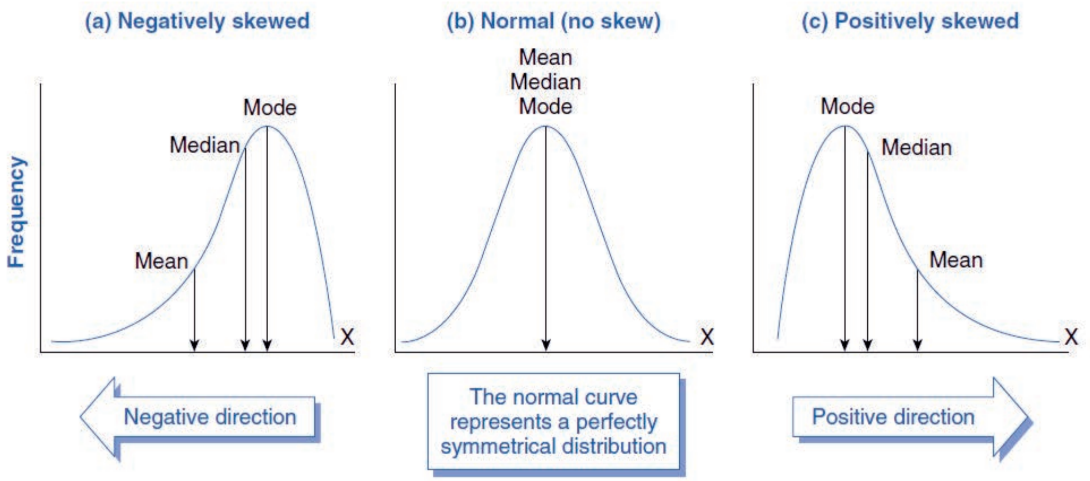


Figure 3.2: Skewed distribution figure

Exercise 3.24

Consider the following frequency distribution:

x	0-20	20-40	40-60	60-80	80-100
f	8	12	30	14	6

a. Draw the bar graph and frequency curve.

b. Calculate mean, median and mode.

c. Describe relationships between the mean, median and mode, and the skewness of the distribution

3.3.2 Measurements of Skewness

Pearson's coefficient of skewness

Activity 3.14

Consider the following data sets:

Data A: $2, 3, 4, 5, 5, 6, 5, 7, 8$

Data B: $2, 3, 1, 4, 8, 5, 8, 6, 8$

1. Calculate the mean, median and standard deviation for each data set.
2. Compare the values obtained in question 1.

In the discussion above, we used the relationships between the measures of central tendency to determine the skewness of a distribution. With the help of central tendencies and standard deviation, it is also possible to determine the skewness of a distribution. This is sometimes called a mathematical measure of skewness. Mathematically, skewness can be measured by calculating:

i. Pearson's coefficient of skewness.

ii. Bowley's coefficient of skewness.

Pearson's coefficient of skewness is obtained by expressing the difference between the mean and the median relative to the standard deviation. It is usually denoted by α.

Pearson's coefficient of skewness is

$$\alpha = \frac{3(\text{mean} - \text{median})}{\text{standard deviation}}.$$

The interpretation of skewness by this approach follows our prior knowledge. From the previous discussion, if the mean = the median, we can see that the distribution is symmetrical. Looking at Pearson’s coefficient of skewness, if the mean is equal to the median then, $\alpha = 0$ so the distribution is symmetrical. Following the same approach, we can state the following interpretation on skewness using Pearson’s coefficient of skewness.

Interpretation

1. If Pearson's coefficient of skewness $\alpha = 0$, then the distribution is symmetrical.
2. If Pearson's coefficient of skewness $\alpha > 0$ (positive), then the distribution is skewed positively (skewed to the right).
3. If Pearson's coefficient of skewness $\alpha < 0$ (negative), then the distribution is negatively skewed (skewed to the left).

Example 1

Calculate Pearson's coefficient of skewness for the data given below and determine the skewness of the distribution.

x	100	105	110	115	120
f	3	4	5	2	1

Solution

$\bar{x} = 108$, $m_d = 110$ and $SD = 5.7$. Thus,

$$\alpha = \frac{3(\text{mean} - \text{median})}{\text{standard deviation}} = \frac{3(108 - 110)}{5.7} = -1.05 < 0.$$

Therefore, the distribution is negatively skewed.

Exercise 3.25

Calculate Pearson's coefficient of skewness from the data below and determine the skewness of the distribution.

x	10	20	30	40	50
f	4	6	5	3	2

Bowley's coefficient of skewness

Previously, you saw how to determine skewness by using relationships between mean, median and standard deviation. It is also possible to determine skewness by using positional measures of central tendency, the quartiles.

A coefficient of skewness that uses quartiles is Bowley's coefficient of skewness, which is usually denoted by β, and is given by:

$$\beta = \frac{Q_3 + Q_1 - 2\,(\text{median})}{Q_3 - Q_1}$$

The interpretation for skewness based on Bowley's coefficient of skewness is also the same as that of Pearson's.

Interpretation

1. If Bowley's coefficient of skewness $\beta = 0$, then the distribution is symmetrical.
2. If Bowley's coefficient of skewness $\beta > 0$ (positive), then the distribution is skewed positively (skewed to the right).
3. If Bowley's coefficient of skewness $\beta < 0$ (negative), then the distribution is negatively skewed (skewed to the left).

Example 2

Find Bowley's coefficient of skewness for the following data and determine the skewness of the distribution.

$11, 8, 7, 13, 12, 14, 15, 17, 20, 16, 19$

Solution

Arranging the data in ascending order, you get:

$7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 20.$

From this arranged data, you can determine the quartiles as:

$Q_1 = 11$, $Q_2 = \text{median} = 14$ and $Q_3 = 17$. Thus,

Bowley's coefficient of skewness is

$$\beta = \frac{Q_3 + Q_1 - 2(\text{median})}{Q_3 - Q_1} = \frac{17 + 11 - 2(14)}{17 - 11} = 0.$$

Therefore, the distribution is symmetrical.

Exercise 3.26

1. Calculate Bowley's coefficient of skewness for the data below and determine the skewness of the distribution.

x	10	20	30	40	50
f	4	6	5	3	2

2. Compare Bowley's coefficient of skewness calculated in question 1 with Pearson's coefficient calculated in Exercise 3.24. Interpret the result.

3.4 Sampling Techniques

Sampling

In Grade 9 and Grade 11, you did some work in statistics including the collection and tabulation of statistical data, frequency distributions and histograms, measures of location (mean, median and mode(s), quartiles, deciles and percentiles), measures of dispersion for both ungrouped and grouped data, and some ideas of probability. In this section, you will study sampling techniques.

Activity 3.15

If you were asked to study something, perhaps you would start by collecting data. Discuss the following questions.

1. Why do you need to collect data?
2. How would you collect the data?
3. From where would you collect the data?

Statistics as a science deals with the proper collection, organization, presentation, analysis and interpretation of numerical data. Since statistics is useful for making decisions or forecasting future events, it is applicable in almost all sciences. It is useful in social, economic and political activities. It is also useful in scientific investigations. Some examples of applications of statistics are given below.

1. Statistics in schools

In schools, teachers rank their students at the end of a semester based on information collected through different methods (exams, tests, quizzes, etc.) which gives an indication of the students' performance.

2. Statistics in business

Statistics is widely used in business to make business forecasts. A successful business must keep a proper record of information in order to predict the future course of the business, and should be accurate in its statistical and business forecasting. Statistics can also be used to help in formulating economic policies and evaluating their effect.

3. Weather forecasting

Statistics is used heavily in the field of weather forecasting. In particular, probability is used by weather forecasters to assess how likely it is that there will be rain, snow, clouds, etc. on a given day in a certain area.

Forecasters will regularly say things like “there is a 90% chance of rain today after 5 pm” to indicate that there’s a high likelihood of rain during certain hours.

4. Sales tracking

Retail companies often use descriptive statistics like the mean, median, mode, standard deviation, and interquartile range to track the sales behavior of certain products.

This gives companies an idea of how many products they can expect to sell during different time periods and allows them to know how much they should keep in stock.

5. Health insurance

Health insurance companies often use statistics and probability to determine how likely it is that certain individuals will spend a certain amount on healthcare each year.

For example, an actuary at a health insurance company might use factors like age, existing medical conditions, current health status, etc. to determine that there’s an

80% probability that a certain individual will spend $10,000 or more on healthcare in a given year.

6. Traffic

Traffic engineers regularly use statistics to monitor the traffic in different areas of a city, which allows them to decide whether or not they should add or remove roads to optimize traffic flow.

Also, traffic engineers often use time series analysis to monitor how traffic changes throughout the day so they can optimize the behavior of traffic lights.

7. Investing

Investors use statistics and probability to assess how likely it is that a certain investment will pay off.

For example, a given investor might determine that there is a 5% chance that the stock of company A will increase 100 times during the upcoming year. Based on this probability, they'll decide how much of their portfolio to invest in the stock.

8. Medical studies

Statistics is regularly used in medical studies to understand how different factors are related. For example, medical professions often use correlation to analyze how factors like weight, height, smoking habits, exercise habits, and diet are related.

If a certain diet and overall weight is found to be negatively correlated, a medical professional may recommend a diet to an individual who needs to lose weight.

9. Manufacturing

Statistics is often used in manufacturing to monitor the efficiency of different processes. For example, manufacturing engineers may collect a random sample of widgets from a certain assembly line and track how many of the widgets are defective.

They may then perform a test to determine if the proportion of widgets that are defective is lower than a certain value that is considered acceptable.

10. Urban planning

Statistics is regularly used by urban planners to decide how many apartments, shops, stores, etc. should be built in a certain area based on population growth patterns.

For example, if an urban planner sees that population growth in a certain part of the city is increasing at high rate compared to other parts of the city then they may decide to prioritize building new apartment complexes in that part of the city.

Note

1. Collection of data is the basis for any statistical analysis. Great care must be taken at this stage to get accurate data. Inaccurate and inadequate data may lead to wrong or misleading conclusions and cause poor decisions to be made.
2. Recall that a population in statistics means the complete collection of items (individuals) under consideration.

It is often impractical and too costly to collect data from the whole population or to make census survey. Consequently, it is frequently necessary to use the process of sampling, from which conclusions are drawn about a whole population. This leads you into an essential statistical concept called sampling which is important for practical purposes.

Definition 3.14

A sample is a limited number of items taken from a population which is being studied or investigated.

A sample needs to be taken in such a way that it is a true representation of the population. It should not be biased so as to cause a wrong conclusion. Avoiding bias requires the use of proper sampling techniques.

Before examining sampling techniques, you need to note the following.

1. **Size of a sample**: There is no single rule for determining the size of a sample of a given population. However, the size should be adequate in order to represent the population.

i. **Homogeneity** or **heterogeneity** of the population: If the population has a homogeneous nature, a smaller size sample is sufficient. (For example, a drop of blood is sufficient to take a blood test from someone).

ii. **Availability of resources**: If sufficient resources are available, it is advisable to increase the size of the sample.

2. **Independence:** Each item or individual in the population should have an equal chance of being selected as a member of the sample.

Exercise 3.27

Your school has 4 sections for grade 12 and the numbers of students are shown in the table below.

Section	Boys	Girls	Total
A	20	30	50
B	20	20	40
C	10	30	40
D	20	10	30
Total	70	90	160

Suppose you want to conduct a survey about the grade 12 students of your school. Since time is limited, you have to select 20 students only.

a. What is the size of the population?

b. What is the size of the sample?

c. Are the following methods of selection acceptable? If not, explain why it is not acceptable.

1) Select 20 girls.

2) Select 20 students from Section A.

3) Select 10 boys and 10 girls.

4) Select 5 students each from the four sections.

Random sampling

There are various techniques of sampling, but they can be broadly grouped into two:

i. Random or probability sampling. **ii.** Non-random or non-probability sampling.

Random or Probability Sampling

Simple random sampling

Simple random sampling is the selection of individuals for which every individual has equal chance of being selected. To apply this method, you can either use the lottery method or a table of random numbers (attached at the end of the textbook).

The lottery method

In this method you need to take the following steps.

Step 1: Prepare slips of paper which are identical in size and color.

Step 2: Write names or code numbers for each member of the population.

Step 3: Fold the slips and put them in a container and mix them up.

Step 4: A blindfold selection are made while blindfolded until a sample of the required size is obtained.

Example 1

A mathematics teacher in a school wants to determine the average height of grade 12 students. There are 8 sections of grade 12 in the school. Assuming that there are 45 students in each class and that she require a sample size of 48 (6 from each section), how can she use the lottery method to select her sample?

Solution

- ✓ Prepare 45 cards of same size and color with number 0 written on 39 of them and the number 1 written on 6 of them.
- ✓ Put the cards on a table with the numbers facing down.
- ✓ Invite the students (one at a time) to come and pick a card.
- ✓ Those who pick cards with the number one on them will be members of the sample.
- ✓ Repeat the same process for each group.

Using a table of random numbers

For this method, you need to use a table of random numbers, and you need to take the following steps.

Step 1: Each member of the population is given a unique number from a set of consecutive numbers.

Step 2: Select arbitrarily one random number from the table of random numbers.

Step 3: Starting with the selected random number, read the consecutive list of random numbers and match these with the members of the population in their consecutive number order.

Step 4: Sort the selected random numbers into either ascending or descending order.

Step 5: If you need a sample of size "n", then select the sample that corresponds with the first "n" random numbers.

Example 2

For the problem in Example 1 above, use the random numbers table attached at the end of the textbook to select a sample of 48 students (6 from each section).

Solution

- ✓ Give each student a roll number from 1 to 45 in alphabetical order.
- ✓ Select arbitrarily one random number from the table of random numbers.
- ✓ From the selected random number, read 45 consecutive random numbers and attach each to the consecutive numbers given to each member of the population.
- ✓ Sort the selected random numbers (together with the numbers from 1 – 45) into ascending or descending order.
- ✓ Take the first 6 random numbers and the corresponding roll numbers. The students whose roll numbers are selected will be part of the sample.

Advantages of simple random sampling

- Lack of bias
- Simplicity
- Less knowledge required

Limitations of simple random sampling

- Difficult to access lists of the full population
- Time consuming
- Cost

Exercise 3.28

1. In your class, there are 20 students whose roll numbers are from 1 to 20. You want to select five students using the random numbers table attached at the end of this textbook. Following the steps explained below, select five students.
   1) Select one random number from the table in a random way.
   2) Read 20 consecutive numbers from the table and attach them to each of the roll numbers as listed.
   3) Sort the random numbers (together with the roll numbers) in ascending order.
   4) Take the first 5 random numbers and the corresponding roll numbers.

Roll Number	Random Number
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Random Number in Ascending Order	Roll Number Rearranged

11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

2. Which roll numbers did you select? Compare the results with others.

Systematic sampling

Systematic sampling is another random sampling technique used for selecting a sample from a population. In order to apply this method, you take the following steps:

If N = size of the population and n = size of the sample, then we use $k = \frac{N}{n}$ for a sampling interval. After this, we arbitrarily select one number between 1 and k, and then every next sample member is selected by considering the k^{th} member after the selected one.

Example 3

In a class, there are 110 students with class list numbers written from 1-110.

You need to select a sample of 10 students. How can you apply the systematic sampling technique?

Solution

You apply the systematic sampling technique as follows:

$N = 110,\ n = 10$ which implies

$$k = \frac{N}{n} = \frac{110}{10} = 11.$$

First, sort the list in ascending order and choose one number at random from the first 11 numbers. If the selected number is 6, then the sample numbers that you obtain by taking every 11^{th} number until you get the tenth sample. The selected sample will be $6, 17, 28, 39, 50, 61, 72, 83, 94, 105$.

Note that in systematic sampling, you use $S_n = S_1 + (n-1)k$ where S_1 is the first randomly selected sample, S_n stands for n^{th} member of a sample and k is the sampling interval.

Advantages of systematic sampling

- Easy to execute and understand
- Control and sense of process
- Clustered selection eliminated
- Low risk factor

Limitations of systematic sampling

- Assumes the size of the population can be determined
- Need for a natural degree of randomness
- Great risk of data manipulation

Exercise 3.29

You want to know how many words on average are listed on one page of a dictionary. The dictionary has 1200 pages in total. You have decided to select 20 pages as a sample for your survey. Supposing you apply the systematic sampling technique, answer the following questions.

a. What is the sampling interval?

b. If you have chosen 13 as the first number, list all the page numbers to be selected as samples.

Stratified sampling

Stratified sampling is useful whenever the population under consideration has some identifiable stratum or categorical difference where, in each stratum, the data values or items are supposed to be homogeneous. In this method, the population is divided into homogeneous groups or classes called strata and a sample is drawn from each stratum. Once you identify the strata, you select a sample from each stratum either by simple random sampling or systematic sampling.

Example 4

If you consider students in a group, you may consider intervals of age as strata. In such a case, you could take the age groups 12-14, 15-17 and 18–20 as stratification of the students. Samples are taken from each stratum proportionally. In this case, strata are age groups from 12-14, 15-17 and 18-20.

Advantages of stratified sampling

- More representative sample
- Great precision
- Administrative convenience

Limitations of stratified sampling

- The success of stratified random sampling depends on effective stratification and appropriate size of the sample to be drawn from each of the stratum. If stratification is faulty, the result will be biased.
- Disproportional stratified sampling requires the assignment of weights to different strata and if the weights assigned are faulty, the resulting sample will not be representative and might give biased results.
- Time consuming and tedious.

Exercise 3.30

In City A, there are 60 secondary schools with 250 classes in total. Suppose that all the sections have 25 boys and 25 girls each. Among the classes, 100 are Social Science stream and 150 are Natural Science stream. You want to select 30 classes from them to conduct a survey about the desired future occupation of the students.

a. In this case, what are the strata?

b. If you have to choose 30 classes as your sample, how many classes you should choose from Social Science stream and Natural Science stream, respectively?

Cluster sampling

Cluster sampling divides the population into subgroups, but each subgroup has similar characteristics to the whole sample. Now, instead of sampling individuals from each subgroup, you randomly select entire subgroups.

Example 5

The list of all agricultural farmers in a village or district may not be easily available but the list of villages or districts is generally available. In this case, every farmer in a sampling unit and every village or district is the cluster.

Advantages of cluster sampling

- Requires fewer resources
- More feasible

Limitations of cluster sampling

- Biased samples
- High sampling error

Figure 3.3 represents the above four sampling techniques.

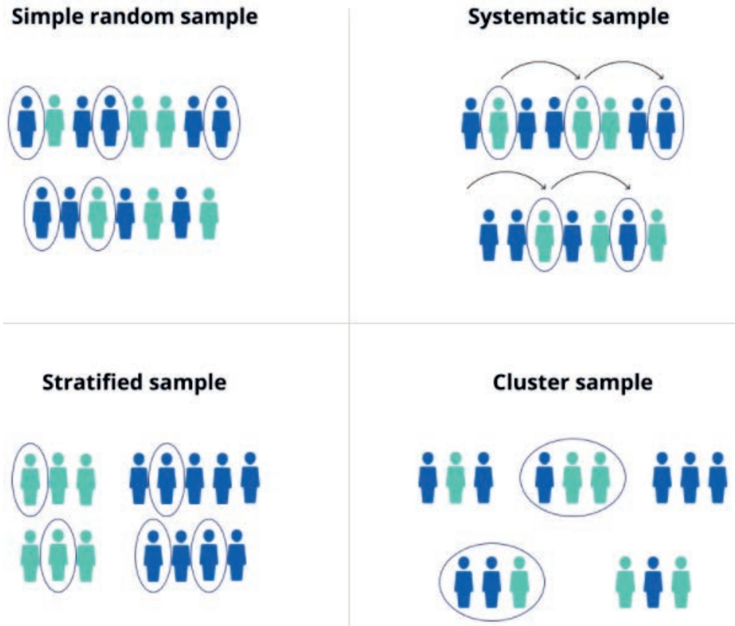


Figure 3.3: Pictorial description of the four sampling techniques

Exercise 3.31

In City B, there are 40 secondary schools with 60,000 students in total. Suppose the City Education Bureau (CEB) has selected 1,000 students from them for survey purposes applying one of the following methods. Name each of the sampling methods.

a. CEB collected students' rosters from all the schools and combined them into one. After assigning a new roll number to each student, CEB selected those numbers of students: 10, 70, 130, 190,. ...

b. CEB collected students' rosters from all the schools and combined them into one. After assigning a new roll number and a random number to each student, CEB rearranged the random numbers in ascending order. CEB selected the first 1,000 students from the list.

c. CEB classified the schools into urban and rural schools. There are 25 urban schools (45,000 students in 900 sections) and 15 rural schools (15,000 students in 500 sections). CEB selected 15 sections from the urban schools and 8 sections from the rural schools to cover about 1,000 students in total.

d. CEB first selected 5 urban schools and 3 rural schools. In each school, CBE further selected Section B of all grades. Since there were 1,600 students in the selected 32 sections, CBE further selected 1,000 students randomly from them.

Multistage Sampling

Multistage sampling is a more complex form of cluster sampling, in which smaller groups are successively selected from large populations to form the sample population used in your study. Due to this multi-step nature, the sampling method is sometimes referred to as phase sampling.

Steps for Carrying out Multistage Sampling Technique

For this method, you need to take the following steps.

Step 1: Define the first sampling frame, assign a unique number to each group in the population, and select a small sample of discrete clusters to form your primary sampling units.

Step 2: Define a second sampling frame for the primary sampling units selected in step 1 and then select random samples from these.

Step 3: If necessary, repeat the previous step.

Step 4: Select the final sample group from the sub-groups using a form of probability sampling, such as simple random sampling or systematic sampling.

Example 6

A research firm in Ethiopia conducted a survey in which it divided the country into its counties and randomly selected some of these counties as a cluster sample (the first stage of sampling).

Each county was then divided into its towns, and areas were chosen at random from each town (the second stage of sampling).

Finally, each town was divided into small areas and households were selected at random from each area. These households formed the sample population for the research study (third stage of sampling).

Advantages of multistage sampling

- Practical for primary data collection for large populations that are geographically dispersed.
- Reduces the costs and time associated with data collection.
- Provides flexibility, as researchers can break down the population as often as necessary to create the sample population they need.
- Allows each stage to use its own sampling method, whether it be stratified sampling, cluster sampling or simple random sampling.

Limitations of multistage sampling

- It introduces a considerable degree of subjectivity, based on the sampling design that surrounds the formation of the sub-groups and their selection.
- To form the sampling frames for multistage random sampling, group-level information is required, sometimes at a national level depending on the target population.
- Typically not as accurate as using simple random sample with the same sample size.

So far, five different sampling techniques have been discussed. It is important to remember that, no one technique is better than the others. Each has its own advantages and limitations.

Exercise 3.32

Suppose, as a statistics officer, you have to survey 1000 households of your Region or City. To select the sample households, you use the multistage sampling method. As the first stage of sampling, you randomly select four Zones or Sub-cities. From each Zone or Sub-city, you further select two Woreda randomly as the second stage sampling.

a. How many households do you have to select from each selected Woreda, if an equal number of households should be selected from each Woreda?

b. Suppose the total numbers of households of the eight selected Woreda are as follows:

100, 300, 300, 400, 500, 600, 800, 1000

If you want to select households proportional to the total number of households in each Woreda, determine the number of sample households to be selected from each Woreda.

Some Statistical Application Software

How to compute the measures of dispersion using microsoft Excel

Range

Excel does not offer a function to compute range. However, we can easily compute it by subtracting the minimum value from the maximum value. The formula would be =MAX( )-MIN( ) where the dataset would be the referenced in both the parentheses. The =MAX( ) and =MIN( ) functions would find the maximum and the minimum points in the data. The difference between the two is the range. The higher the value of the range, the greater is the spread of the data.

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Average	=AVERAGE(C3:C6)	=AVERAGE(D3:D6)
Range	=MAX(C3:C6)-MIN(C3:C6)	=MAX(D3:D6)-MIN(D3:D6)

Variance

The calculation of variance differs slightly depending on whether the data set describes a sample or the entire population. We have already seen that variance is nothing but the average of the squared deviations. When we are computing the variance for a population, we divide the sum of squared deviations by n. However, when we compute the variance for a sample, we divide the sum of squared deviations by (n - 1).

This change is taken care of by Excel with two different functions: =VAR.P( ) for the population variance, and =VAR.S( ) for the sample variance.

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Average	=AVERAGE(C3:C6)	=AVERAGE(D3:D6)
Population Variance	=VAR.P(C3:C6)	=VAR.P(D3:D6)
Sample Variance	=VAR.S(C4:C7)	=VAR.S(D4:D7)

If we treat our data set as the population, then the variance for Arun is 1275, and the variance for John is 162.5. If we treat our data as a sample, the variance for Arun is 1189.58, and the variance for John is 50.

Older versions of Excel used =VARP( ) and =VARS( ) to calculate population variance, and sample variance, respectively.

Microsoft Excel also supports two other functions that calculate variance, =VARA( ) for sample variance, and =VARPA() for population variance. These differ from the other variance functions in how they treat certain text strings within the data.

=VARA( ) and =VARPA( ) can handle the following text strings that =VAR.S( ) and =VAR.P( ) ignore:

1. Logical values such as TRUE and FALSE are counted, and treated as 1, and 0, respectively.
2. Any text value is counted, and is treated as 0.

Standard Deviation

We already know that the standard deviation is nothing but the square root of variance. Naturally, if the variance computation is different for a sample and for a population, the standard deviation would be different as well. Similar to variance, Excel offers two functions, =STDEV.S( ) for sample standard deviation, and =STDEV.P( ) for population standard deviation.

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Average	**=AVERAGE(C3:C6)**	**=AVERAGE(D3:D6)**
Population SD	**=STDEV.P(C3:C6)**	**=STDEV.P(D3:D6)**
Sample SD	**=STDEV.S(C3:C6)**	**=STDEV.S(D3:D6)**

Older versions of Excel support =STDEV() for sample standard deviation, and =STDEVP() for population standard deviation.

Standard deviation can also be computed on logical strings, and text, just like variance. The function for sample variance is =STDEVA( ) and the function for population variance is =STDEVPA( ), if the text values are to be counted. The treatment of the text values remain the same as with =VARA( ) and =VARPA( ) functions.

Average Absolute Deviation

To avoid the problems associated with squaring the quantities with dimensions, we may want to check dispersion using the average of absolute values of the deviations. Excel includes a function for it as well. It can be computed with =AVEDEV( ).

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Average	**=AVERAGE(C3:C6)**	**=AVERAGE(D3:D6)**
Average Dev.	**=AVEDEV(C3:C6)**	**=AVEDEV(D3:D6)**

=AVEDEV( ) would ignore any text values, or logical values. However, cells with 0 are counted, and treated as numbers.

Inter-Quartile Range (IQR)

Microsoft Excel has two functions to compute quartiles. The inter-quartile range has to be calculated as the difference between the quartile 3 and quartile 1 values.

Quartiles can be calculated using =QUARTILE.INC( ) or =QUARTILE.EXC( ). Both functions calculate the quartiles by calculating the percentiles on the data. However, the =QUARTILE.EXC calculates exclusive quartiles, and cannot calculate quartile 0 or quartile 4 (the extreme values are excluded). The inclusive function =QUARTILE.INC( ) can be used to calculate quartiles including quartile 0 and quartile 4.

Both functions have the following syntax: =QUARTILE.INC(range, quartile number) where the quartile number can be between 0 and 4. Any quartile number outside these values would return an error.

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Quartile 3	**=QUARTILE.INC(C3:C6,3)**	**=QUARTILE.INC(D3:D6,3)**
Quartile 1	**=QUARTILE.INC(C3:C6,1)**	**=QUARTILE.INC(D3:D6,1)**
IQR	**=C7-C8**	**=D7-D8**

Max. Score = 100	Arun	John
Math	100	45
Physics	40	65
Chemistry	20	70
Programming	100	80
Quartile 3	**=QUARTILE.EXC(C3:C6,3)**	**=QUARTILE.EXC(D3:D6,3)**
Quartile 1	**=QUARTILE.EXC(C3:C6,1)**	**=QUARTILE.EXC(D3:D6,1)**
IQR	**=C7-C8**	**=D7-D8**

In both the above examples, Excel would calculate the quartile values by extrapolation because there are not enough data points. The quartile boundaries would lie between two values in our data set.

Older versions of Excel had a single function for quartile, =QUARTILE() and that was identical to the =QUARTILE.INC() function in the current versions. There was no equivalent of =QUARTILE.EXC() function in the earlier versions.

Summary

1. Measures of dispersion help to decide the degree of variability.
2. There are two types of measure of dispersion.
   i. Absolute and ii. Relative.
3. Range = $L - S$.
4. Inter-quartile range = $IR = Q_3 - Q_1$.
5. Mean deviation about the mean = $MD(\bar{x}) = \frac{\sum|x-\bar{x}|}{n}$.
6. Mean deviation about the median = $MD(m_d) = \frac{\sum|x-m_d|}{n}$.
7. Mean deviation about the mode = $MD(m_o) = \frac{\sum|x-m_o|}{n}$.
8. Quartile deviation = $QD = \frac{Q_3-Q_1}{2}$.
9. Variance = $\sigma^2 = \frac{\sum f(x-\bar{x})^2}{\sum f}$.
10. Standard deviation = $\sigma = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$.
11. Coefficient of range = $CR = \frac{L-S}{L+S}$.
12. Coefficient of quartile deviation = $CQD = \frac{Q_3-Q_1}{Q_3+Q_1}$.
13. Coefficient of mean deviation = $CMD = \frac{MD}{\bar{x}}$.
14. Coefficient of variation = $CV = \frac{\sigma}{\bar{x}} \times 100$.
15. For a given distribution:
    - If $\bar{x} > m_d > m_o$, then the distribution is **positively skewed.**
    - If $\bar{x} < m_d < m_o$, then the distribution is **negatively skewed.**
    - If $\bar{x} = m_d = m_o$, then the distribution is **symmetrical**.
16. Pearson's coefficient of skewness $= \alpha = \frac{3(\text{mean}-\text{median})}{\text{standard deviation}}$
    - If $\alpha > 0$, then the distribution is **positively skewed**.
    - If $\alpha < 0$, then the distribution is **negatively skewed**.
    - If $\alpha = 0$, then the distribution is **symmetrical.**

17. Bowley's coefficient of skewness= $\beta = \frac{Q_3+Q_1-2(\text{median})}{Q_3-Q_1}$

- If $\beta > 0$, then the distribution is **positively skewed**.
- If $\beta < 0$, then the distribution is **negatively skewed**.
- If $\beta = 0$, then the distribution is **symmetrical.**

18. A **population** is the complete set of items which are of interest in any particular situation.
19. It is not possible to collect information from the whole population because it is costly in terms of time, energy and resources. To overcome these problems, we take only a certain part of the population called a **sample.**
20. A sample serves as representative of the population, so that we can draw conclusions about the entire population based on the results obtained from the sample.
21. There are two methods of sampling:
    i. The random (probability) sampling method.
    ii. The non-random (non-probability) sampling method.
22. In random sampling, every member of the population has an equal chance of being selected.

Review Exercise

1. Find the range, inter-quartile range, coefficient of range and coefficient of quartile deviation of the following data set.

Class	15-20	20-25	25-30	30-35
f	8	21	15	4

2. Find the mean deviation about the mean of mass (in kg) and the coefficient of mean deviation about the mean of 460 infants born in hospital in one year from the following table.

Mass	2-2.5	2.5-3	3-3.5	3.5-4	4-4.5	4.5-5
f	17	97	187	135	18	6

3. Find variance and standard deviation of the data which represents the marks scored by ten students in a class test:

   $30, 29, 25, 33, 35, 38, 37, 48, 40, 44.$

4. Consider the data given below:

   $$4, 8, 7, 10, 11$$

   a. Find the standard deviation for the data.
   b. Add 10 to all the values then find the standard deviation of the new values.
   c. Multiply all the values by 5 then find the standard deviation of the new data.
   d. Compare the results you obtained in parts b and c with the results you obtained in part a and write a conclusion.

5. Calculate the coefficient of variance of the amount of money (in Birr) a group of children spent on food during a school trip .The amounts spent are:

   $$10, 15, 5, 20, 25, 30, 35, 40.$$

6. 53 students were asked to write the total number of hours per week they spent to studying mathematics. With this information find the mean deviation about the mode, the standard deviation and the coefficient of variation.

x	4	6	8	10	12
f	8	21	15	4	5

7. The mean and standard deviation of 15 observations are found to be 10 and 5, respectively. On rechecking it was found that one of the observations with the value 8 was incorrect. Calculate the mean and standard deviation if the value of the corrected observation was 21.
8. Suppose that each day laboratory technician A completes 40 analyses with standard deviation of 5. Technician B completes 160 analyses with standard deviation 15. Which employee shows less variation?
9. Find the range, mean, variance and standard deviation of the first n natural numbers.
10. A wall clock strikes the bell once at 1 o'clock, 2 at 2 o'clock and 3 at 3 o'clock and so on. How many times will it strike in a particular day? Find variance and standard deviation of the number of strikes the bell makes a day on the hour.
11. 1500 visitors turned up to the city library during the first month of the academic year. They who borrowed 6750 books altogether. Find and interpret the coefficient of variation if the square of the sum of the borrowed books is 33315.
12. The following data obtained from the police records of the accident cases in Dire Dawa city police station from February 2009-August 2013 G.C.

X(per month)	*variance*	*Standard Deviation*
Death	21.20	3.83
Heavy injury	47.20	12.85
Light injury	88.40	12.70
No. of accident	276.40	12.80

a. Determine the coefficient of variation of death, heavy injury, light injury and number of accidents.

b. Based on the values obtained in part (a) give your interpretation of the data.

13. The mean and coefficient of variation of seven observations are 8 and 50, respectively. If five of those are 12, 14, 10, 4 and 2, then find the remaining two observations.

14. For a group of 100 students the mean and coefficient of variation of their marks were found to be 60 and 25, respectively. Later on it was found that the scores 45 and 70 were wrongly entered as 40 and 27. Find the corrected mean and coefficient of variation.

15. If in the following distribution the mean deviation about the mean and coefficient of mean deviation about the mean are 180 and 10, respectively, then find the missing frequency f.

x	11-13	13-15	15-17	17-19	19-21	21-23	23-25
f	7	6	9	13	F	5	4

16. The mean deviation about the median and coefficient of mean deviation about the median of the following data are 440 and 8, respectively. Find the values of s and t if the total frequency is 82.

x	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
f	5	17	2	12	s	2	7	t	9	4

UNIT 4

INTRODUCTION TO LINEAR PROGRAMMING

Unit Outcomes

By the end of this unit, you will be able to:

- Deduce how to find regions of inequality graphs.
- Solve systems of linear inequality.
- Construct linear programing problems.
- Solve real life problems of linear programming problems.
- Use spreadsheet to solve Linear Programming Problems.

Unit Contents

KEY Words

- Bounded solution region
- Constraints
- Decision variables
- Equation of a line
- Feasible region
- Half planes solution
- Minimum value
- Maximum value
- Objective function
- Optimal value
- Slope of a line
- Vertex (corner point)

Introduction

In the previous grades, you discussed systems of linear equations and their applications in solving the day-to-day problems. In Grade 9, you studied about linear inequalities and system of linear inequalities in two variables and their solutions by graphical method.

In this unit, you shall study some linear programming problems and their solutions only by graphical method though there are many other methods to solve such kinds of problems. As part of its components, the unit introduces different real life application problems of Linear Programming Problems. Finally, you will practice how to solve linear programing problems by the help of spreadsheet.

Application of the Unit as a career in Mathematics - software engineering: A software engineer creates, writes, develops and interprets different computer programs. Software engineers use mathematics to write a program and to solve various computing problems. If you are interested in a career in software engineering, therefore, you should study these mathematical subjects: algebra, linear programming, statistics and calculus.

4.1 Graphical Solutions of System of Linear Inequalities

Activity 4.1

1. Plot the linear equation of the form $y = mx + b$ for different values of m and b. For instance, $m < 0, m > 0$ and $m = 0$.
2. Find slope and y-intercept of the following:
   a. $y = 3x + 2$
   b. $y = \frac{1}{3}x + 2$
   c. $y = -4x + 5$
   d. $x + 2y = 2$
3. Draw the graph of the following:
   a. $y = 2$
   b. $x = -4$
   c. $y = -3$

In Unit 4 of Grade 9 Textbook, you studied about inequalities and how their solutions are determined. In this section, you will study graphical solutions of linear inequalities in one and two variables. You will also learn about how to translate word problems into mathematical expression. Here we get certain statements involving a sign "<" (less than), ">" (greater than), "≤" (less than or equal) and" ≥" (greater than or equal) which are known as inequalities.

In the process, you recognize that the study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, economics, finance, psychology, and so on.

4.1.1 Inequalities

Let us consider the following situations:

Alem went to market with Birr 500 to buy wheat flour which was available in packets of 5kg. The price of one packet of wheat flour was Birr 130. If x denotes the number of packets of wheat flour she bought, then the total amount of Birr she spent would be $130x$. Since, she had to buy wheat flour in packets, she would not be able to spend the total amount of Birr 500. Explain.

Clearly the statement is not an equation as it does not involve the sign of equality.

$$\text{Hence, } 130x < 500 \qquad (1)$$

Consider the inequality (1), $130x < 500$ clearly, where x can never be fraction or negative. Left hand side of this inequality is $130x$ and the right hand side is 500.

When $x = 0$, i.e. $130 \times 0 = 0 < 500$, the statement is true.

When $x = 1$, i.e. $130 \times 1 = 130 < 500$, the statement is true.

When $x = 2$, i.e. $130 \times 2 = 260 < 500$, the statement is true.

When $x = 3$, i.e. $130 \times 3 = 390 < 500$, the statement is true.

When $x = 4$, i.e. $130 \times 4 = 520 < 500$, the statement is false.

In the above situation, you find that the values of x, which make the above inequality a true statement, are 0, 1, 2, 3. These values of x, which make the inequality a true statement, are called solutions of inequality and the collection of these numbers, {0, 1, 2, 3} is called the solution set.

Thus, any solution of an inequality in one variable is a value of the variable that makes it a true statement. However, the solutions of the inequality shown above is by *trial and error* which is not very efficient because, it is time consuming and sometimes not feasible. Thus, you must have more efficient systematic techniques for solving inequalities. Before looking at these techniques, you should go through some more properties of numerical inequalities and follow them as rules, while solving the problems in inequalities.

You will recall the following rules while solving linear equations:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation.

Rule 2 Both sides of an equation, may be multiplied (or divided) by the same non-zero number.

Nevertheless, Rule 2 has an exception when the operation involves negative number. The sign of inequality is reversed whenever you multiply (or divide) both sides of an inequality by a negative number.

Consider the following rules for solving an inequality:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.

Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of the inequality is reversed.

Example 1

Solve $7x + 3 < 5x + 9$ and, then, show the graph of the solution on a number line.

Solution

By Rule 1, we subtracted 3 and $5x$ from both sides of the inequality.

$$7x + 3 < 5x + 9,$$
$$7x - 5x < 9 - 3,$$
$$2x < 6,$$
$$x < 3.$$

Which is equivalent to $(-\infty,\ 3)$, the graphical representation of the solution is given in Figure 4.1.

Figure 4.1

Example 2

Solve the following problem and then, show the graph of the solution on a number line.

$$\begin{cases} 2x - 3 < 1 \\ 4x - 1 \geq 1 \end{cases}$$

Solution

From the first, $2x < 4$, $x < 2$. From the second, $4x \geq 2, x \geq \frac{2}{4}$, $x \geq \frac{1}{2}$. Then, the answer is $\frac{1}{2} \leq x < 2$.

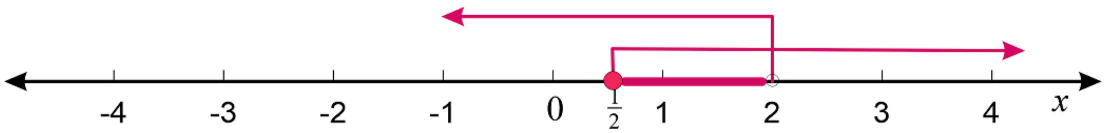


Figure 4.2

Example 3

Solve $\frac{3x-5}{2} \geq \frac{x+1}{4} - 2$ and show the graph of the solution on a number line.

Solution

By Rule 2

$$\frac{3x-5}{2} \geq \frac{x+1}{4} - 2$$
$$\frac{3x-5}{2} \times 4 \geq \frac{x+1}{4} \times 4 - 2 \times 4$$
$$2(3x-5) \geq x + 1 - 8$$
$$6x - 10 \geq x - 7$$
$$5x \geq 3$$
$$x \geq \frac{3}{5}$$

That is, the interval $[\frac{3}{5}, \infty)$ whose graphical representation of the solution is given in Figure 4.3.

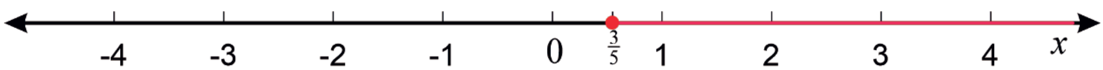


Figure 4.3

Exercise 4.1

Solve the following inequalities and show their solutions on number lines.

a. $5x + 1 < 2x + 7$

b. $\begin{cases} 3x + 1 < 16 \\ -2x + 5 \leq 13 \end{cases}$

c. $-1 < -2x + 3 \leq 2$

d. $\frac{2x}{3} - 3 > \frac{16x}{21} - \frac{13}{3} - \frac{2x}{15}$

e. $\frac{x-4}{2} < \frac{7x}{2} - (3x + 2)$

f. $5(x - 4) + 17 - 14x > 13 - 4x$

g. $12x - 1 \leq 3(4x - 3)$

h. $11(2x - 15) < x + 3$

i. $\frac{x}{x-3} < 2, x \neq 3$

j. $\frac{x+2}{x} - 1 \geq 0,\ x \neq 0.$

4.1.2 Word Problem of Linear Inequality

To solve the word problem related to real world situations you can use the following procedures:

Step 1: Understand the problem

Step 2: Setting up the model

Step 3: Solving the problem

Example 1

A student has Birr 120 and wants to buy some exercise books and pens. The cost of one exercise book is Birr 20 and that of a pen is Birr 10. Write the mathematical expression.

Solution

Let x denotes the number of exercise books and y denotes the number of pens that the student will buy, then the total amount that will be spent by the student is Birr $(20x + 10y)$. Then, we have:

$$20x + 10y \leq 120. \qquad (2)$$

In this case the total amount spent may be up to Birr 120. Note that expression (2) consists of two statements

$$20\,x + 10\,y < 120, \quad (3)$$

$$20\,x + 10\,y = 120. \quad (4)$$

Accordingly, expression (3) is not an equation, i.e., it is an inequality while statement (4) is an equation. Statement, $130\,x < 500$ is an example of linear inequalities in one variable and (2) is an example of linear inequalities in two variables.

Some more examples of inequalities are:

$$ax + b < 0, \quad (5)$$

$$ax + b > 0, \quad (6)$$

$$ax + b \geq 0, \quad (7)$$

$$ax + by \leq 0, \quad (8)$$

$$ax + by \geq 0. \quad (9)$$

Inequalities (5) and (6) are strict inequalities while inequalities (7), (8) and (9) are slack inequalities. Besides, inequalities from (5) to (7) are linear inequalities in one variable x when $a \neq 0$, while inequalities (8) and (9) are linear inequalities in two variables x and y when $a \neq 0$, and $b \neq 0$.

Example 2

A taxi charges a flat rate of Birr 1.75, plus an additional Birr 0.75 per km. If Selam has at most Birr 10 to spend on the car ride, how far could she travel?

Solution

The total cost to travel x distance is T, $T = 0.75x + 1.75$. She has only Birr 10.00. Therefore, $T = 0.75x + 1.75 \leq 10$. Solving for x, you obtain $x \leq 11$. Selam would travel a maximum distance of 11km.

Example 3

Find all pairs of consecutive odd natural numbers both of which are larger than 20, and their sum is less than 80.

Solution

Let n be any natural number, then $2n-1$ and $2n+1$ are two consecutive odd natural numbers. From the given information

$$2n-1>20, \quad (1)$$

$$(2n-1)+(2n+1)<80. \quad (2)$$

Simplifying (2), we get $4n<80$,

$$\text{i.e. } n<20. \quad (3)$$

From (1), $2n>21$,

$n>\frac{21}{2}$.

Combining (1) and (3), we obtain, $10.5<n<20$, thus, n takes the values, 11, 12… 18, 19. So, the required possible pairs will be (21, 23), (23, 25), (25, 27), (27, 29), (29, 31), (31, 33), (33, 35), (35, 37), (37, 39).

Exercise 4.2

1. To get an "A" grade in a course, one must obtain an average of 85 marks or more in five examinations (each out of 100 marks). If Semira's marks in the first four examinations are 87, 82, 81 and 80, then, find the minimum marks that Semira must receive in the fifth examination to obtain "A" grade in the course.
2. Find all pairs of consecutive odd positive integers both of which are smaller than 20 and their sum is more than 21.

3. Show that the following word problems are expressed by inequalities not by equation.
   a. Lenssa practices her bicycle for at least 12 hours per week. She practices for $3/4$ of an hour each session. If Lenssa has already practiced 3 hours this week, how many more sessions remain for her to meet or exceed her weekly practice goal?
   b. The base of a rectangular prism has a length of 13 cm and a width of $1/2$ cm. The volume of the prism is less than or equal to 65cm^3. Find all possible heights of the prism.

4.1.3 Linear Inequalities in Two Variables and Their Graphical Solutions

In the previous section, you have seen that an inequality in one variable is a visual representation and is a convenient way to represent the solutions of the inequality. Now, you will discuss the graph of a linear inequality in two variables.

Activity 4.2

Choose any three points and show in which side of the line $y = x$, in the xy-coordinate plane.

You know that a line divides the Cartesian plane into two parts. Each part is known as a half plane. A vertical line will divide the plane into left and right half planes and a non-vertical line will divide the plane into lower and upper half planes.

Definition 4.1

Half-Plane: The region on a side of a line in the xy-plane.

Vertical Line Non-vertical Line

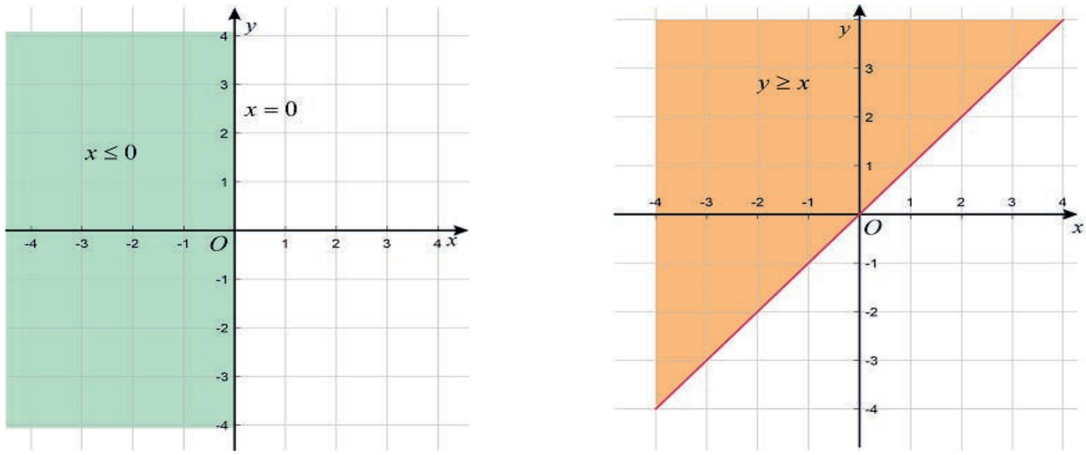


Figure 4.4

A point in the Cartesian plane will either lie on a line or on either of the half planes I or II.

Steps to draw the graph of inequalities

1. Change the inequality sign to an equal sign, and then plot the line.
   a. If an inequality is of the form $y > mx + c$ or $y < mx + c$, then the points on the line $y = mx + c$ are not included in the solution region. So, draw a broken or dotted line,
   b. If an inequality is of the type $y \geq mx + c$ or $y \leq mx + c$, then the points on the line $y = mx + c$ are also included in the solution region. So, draw a solid line.
2. Test a point created in one half-plane.
   a. If the point in the half-plane satisfies the inequality, then the entire half-plane satisfies the inequality.
   b. If the point does not satisfy the inequality, then the entire half-plane does not satisfy the inequality.
3. If a point from one half plain fails to satisfy, test a point from the other half-plane.
4. Shade in any half-planes that satisfy the inequality.

Example 1

Graph the solution set for the following inequalities:

a) $y < 5x + 7$ b) $y \geq 2x + 3$

Solution

a) First change the inequality to the equation, $y = 5x + 7$. It is the line that passes through $(0, 7)$ and its slope is 5. Note that the line is dashed (broken) since the inequality is <, the points on the line are not included in the region. Secondly, test the point (0, 0).

$$(LHS) = 0,$$

$$(RHS) = 5 \times 0 + 7 = 7.$$

$$(LHS) < (RHS)$$

Thus, this point satisfies the inequality. Therefore, every point in the lower side of the line $y = 5x + 7$ satisfies the inequality. Finally, shade the region in the half-plane to the right of the line.

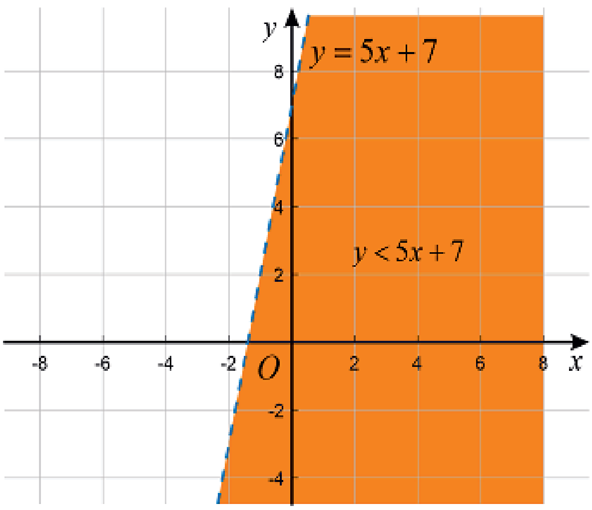


Figure 4.5

b) To plot the region of the solution, first draw the graph of $y = 2x + 3$, whose slope is 2 and y-intercept is $(0, 3)$.

To identify the region test the point $(0, 0)$,

$$(LHS) = 0,$$

$$(RHS) = 2 \times 0 + 3 = 3$$

$$(LHS) < (RHS).$$

Thus, this point does not satisfy the inequality. So the region should be on the other side as shown in the Figure 4.6. Note that the line is solid since the inequality sign is $\geq$. This means that any point on the line satisfy the given inequality.

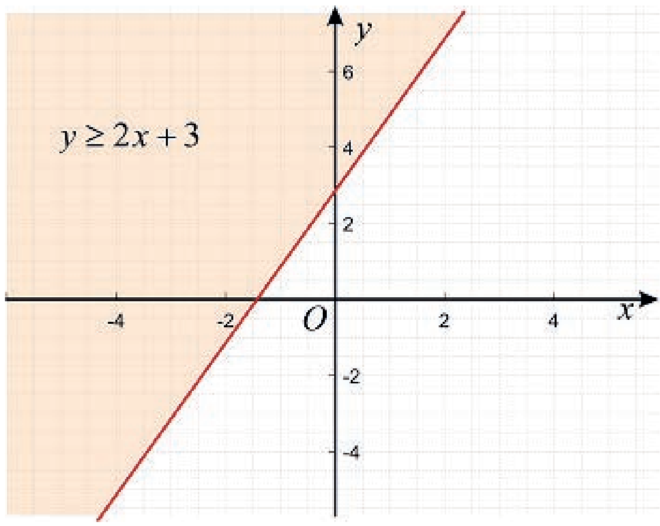


Figure 4.6

Example 2

Write and graph an inequality that describes all ordered pairs whose y-coordinate is at least 5.

Solution

For this problem the value of x is all real numbers. But the y value is any number greater than or equal to 5, i.e., $y \geq 5$ is the required inequality. The graphical representation is shown in Figure 4.7.

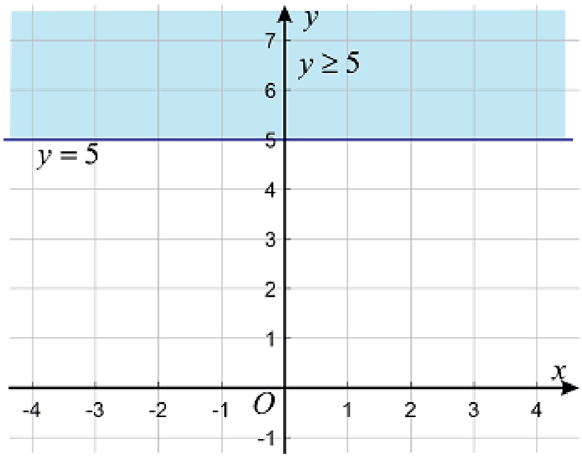


Figure 4.7

Exercise 4.3

1. Graph the solution set for the following inequality:

   a. $y > 5x + 3$ b. $y \leq -\frac{1}{2}x + 7$

   c. $x > 4$ d. $5x + 2y > 6$

2. Is $(2, 5)$ a solution of the inequality $y + 5x \geq 1$?

3. Determine whether or not $(2, \frac{1}{2})$ is a solution to $5x - 2y < 10$.

4. Write an inequality that describes all points in:

   a. the upper half-plane above x-axis;

   b. the lower half-plane below x-axis;

   c. the half-plane left of y-axis;

   d. the half-plane right of y-axis.

4.1.4 Systems of Linear Inequalities in Two Variables and Graphical Solutions

A system of linear inequalities is a collection of two or more linear inequalities to be solved simultaneously. A graphical solution of a system of linear inequalities is the graph of all ordered pairs (x, y) that satisfy all the inequalities. Such a graph is called the solution region (or the feasible region).

Activity 4.3

A student wants to spend at most Birr 15.00 for his breakfast milk and bread. Each cup of milk costs Birr 2.40 and each loaf of bread costs Birr 1.20. Write an inequality that represents the number of cups of milk (x) and loaf of bread (y) a student can buy on his budget.

How to Solve a System of Linear Inequalities in Two Variables

1. Using the technique of graphing inequalities, graph all of the given inequalities.
2. If the inequality is"$y \geq$ ….” or “ $>$ ….”, then you shadow the upper side of the line. Similarly, if the inequality is"$y \leq$ ….", or " $y <$ ….", then you shadow the lower side of the line.
3. The overlapped shaded region is the solution of the system.
4. Regarding the intersection of the lines,
   a. it is part of the solution if all given inequalities include the lines (i.e., $\leq$ or $\geq$)
   b. it is not part of the solution if any one of the given inequalities does not include the line (i.e., $<$ or $>$).

Example 1

Find a graphical solution to the system of linear inequalities $\begin{cases} y \geq -x + 3 \\ y \leq 2x \end{cases}$

Solution

First draw the lines $y = -x + 3$ and $y = 2x$ in the same coordinate plane.

For the first inequality, shade the region above the line $y = -x + 3$ and for the second shade the region below the line $y = 2x$.

When solving the system $\begin{cases} y = -x + 3 \\ y = 2x \end{cases}$ simultaneously we arrive at the point of intersection.

Thus, the overlapped shaded region (green part) is the region that meets the given system of linear inequalities. In this example, the point $(1, 2)$ is also part of the solution.

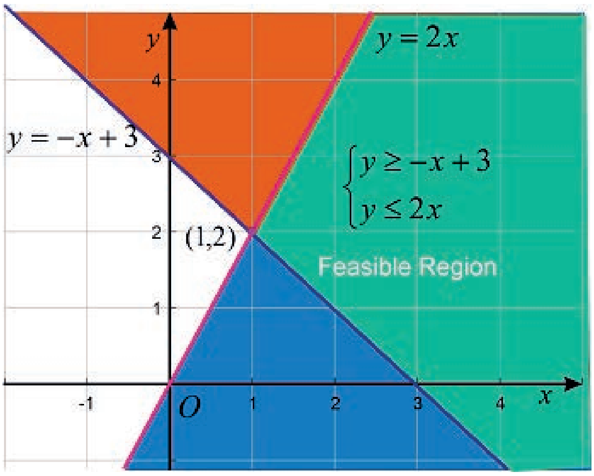


Figure 4.8

Example 2

Find a graphical solution to the system of linear inequalities.

$$\begin{cases} 5x - y > -7 \\ 2x + y \geq 14 \end{cases}$$

Solution

First solve the inequality for y.

$$\begin{cases} y < 5x + 7 \\ y \geq -2x + 14 \end{cases}$$

Then, draw the lines on the same coordinate.

$$\begin{cases} y = 5x + 7 \\ y = -2x + 14 \end{cases}$$

The first inequality is $<$, so shade the lower side of the dot line. The second one is $\geq$, hence, shade the upper side of the solid line. Thus, the overlapped shaded region (orange part) is the region that meets the given system of linear inequalities.' To find the point of intersection of the two lines, solve

$\begin{cases} y - 5x = 7 \\ y + 2x = 14 \end{cases}$, simultaneously, and obtain the point (1, 12).

Which is not part of the solution region.

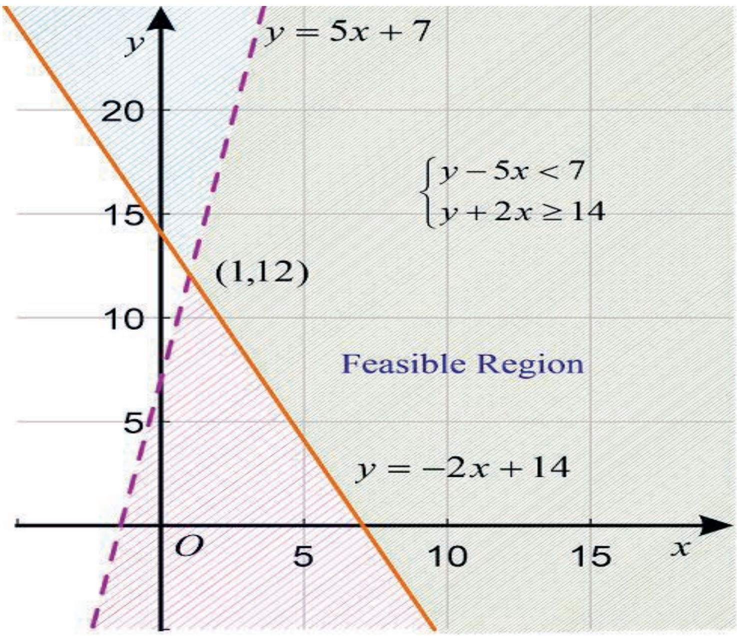


Figure 4.9

Example 3

Determine whether or not the order pairs $(0, 0), (2, 3), (-8, 2)$,and $(-4, 6)$ are in the solutions of the system

$$\begin{cases} x + 3y > 3 \\ -x + y \leq 6 \end{cases}$$

Solution

To test whether or not each point is in the solution region of the system, we use the following table:

Order pairs	Satisfies $x + 3y > 3$	Satisfies $-x + y \leq 6$	Satisfies both inequalities
(0, 0)	No	Yes	No
(2, 3)	Yes	Yes	Yes
(-8, 2)	No	No	No
(-4, 6)	Yes	No	No

Therefore, the only point in the solution region is (2, 3).

Exercise 4.4

1. Solve the following systems of inequalities graphically.

a. $\begin{cases} y \leq 2x + 2 \\ y \geq -2x + 6 \end{cases}$ b. $\begin{cases} 2x + y < 8 \\ x + y \geq 1 \end{cases}$ c. $\begin{cases} y - 2x \leq 4 \\ 2x + y \geq 4 \end{cases}$

d. $\begin{cases} x - y < 0 \\ x + 2y > 3 \end{cases}$ e. $\begin{cases} 2x + 3y \geq 6 \\ x + 5y \leq 4 \end{cases}$ f. $\begin{cases} y - x \geq 0 \\ 2x - 3y \leq 1 \end{cases}$

g. $\begin{cases} y + x > 0 \\ y - 2x < 5 \end{cases}$

2. Determine whether or not the ordered pair is a solution to the system $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a. $(-2, 4)$ b. $(3, 1)$ c. $(1, 3)$ d. $(0, 0)$

4.1.5 Further on the System of Inequalities

Definition 4.2

A point of intersection of two or more boundary lines of a solution region is called a vertex (or a corner point) of the region.

Example 1

Solve the following system of inequalities graphically.

$$\begin{cases} 5x + 4y \le 40 \\ x \ge 1 \\ y \ge 2 \end{cases} \quad (1)$$

Solution

We first rewrite the first inequality, $y \le -\frac{5}{4}x + 10$

Draw the graphs of the lines,

$$y = -\frac{5}{4}x + 10 \quad (2)$$

$$x = 1 \quad (3)$$

$$y = 2 \quad (4)$$

Then, we know that the inequality $y \le -\frac{5}{4}x + 10$ represents the shaded region below the line $y = -\frac{5}{4}x + 10$ and the second inequality represents the shaded region right of the line $x = 1$ but third inequality represents the shaded region above the line $y = 2$.

To find the vertices of the region, solve the equation (2) and (3), (2) and (4) simultaneously.

For (2) and (3), substituting $x = 1$ into (4), $y = -\frac{5}{4} \times 1 + 10 = \frac{35}{4}$

Therefore, $(x, y) = (1, \frac{35}{4})$

Similarly, for (2) and (4), substituting $y = 2$ into (2),

$$2 = -\frac{5}{4}x + 10,$$

$$\frac{5}{4}x = 8,$$

$$x = \frac{32}{5}.$$

Therefore, $(x, y) = (\frac{32}{5}, 2)$

Hence, the shaded region including all the points on the lines are also the solution of the given system of the linear inequalities.

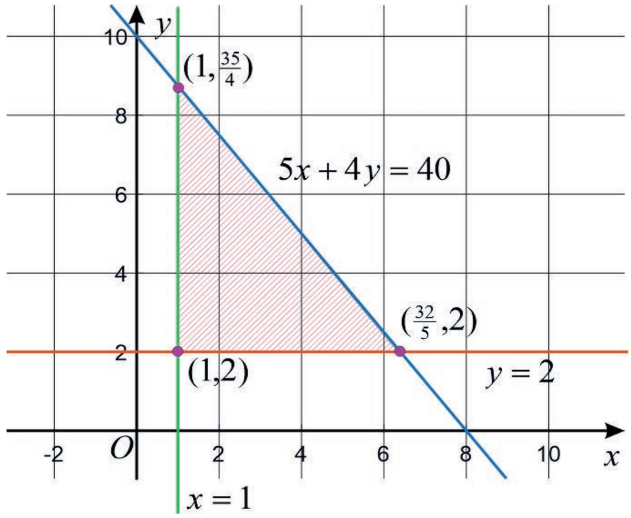


Figure 4.10

In many practical situations involving system of inequalities, the variables x and y often represent quantities that cannot have negative values. For example, number of units produced, number of articles purchased, and number of hours worked, etc. Clearly, in such cases, $x \geq 0, y \geq 0$ and the solution region lies only in the first quadrant.

Example 2

Ficho needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 40 more than the number of pencils. The pencils cost Birr 2.00 and the answer sheet cost Birr 1.00. Ficho's budget for these supplies allows for a maximum cost of Birr 400.

a. Write a system of inequalities to model this situation.

b. Graph the system.

c. Can Ficho purchase 100 answer sheets and 150 pencils?

d. Can Ficho purchase 150 answer sheets and 100 pencils?

Solution

a. Letting x represents the number of answer sheets and y is the number of pencils. $\begin{cases} x \geq y + 40 \\ x + 2y \leq 400 \\ x \geq 0 \\ y \geq 0. \end{cases}$

Re-write the inequalities:

$$\begin{cases} y \leq x - 40 \\ y \leq -\frac{1}{2}x + 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Then, draw the lines of equations:

$$\begin{cases} y = x - 40 & (1) \\ y = -\frac{1}{2}x + 200 & (2) \\ x = 0 & (3) \\ y = 0 & (4) \end{cases}$$

To find the three vertices, solve each pair of equations simultaneously.

For (1) and (2),

$$x - 40 = -\frac{1}{2}x + 200, \quad \frac{3}{2}x = 240, \quad x = 240 \times \frac{2}{3} = 160$$

$$y = 160 - 40 = 120.$$

Therefore, $(x, y) = (160,120)$. The other two vertices are also found by solving (1) and (4), and (2) and (4) simultaneously, (40, 0) and (400, 0).

b. The graph is as shown in Figure 4.11.

c. Ficho cannot purchase 100 answer sheets and 150 pencils.

d. Ficho can purchase 150 answer sheets and 100 pencils.

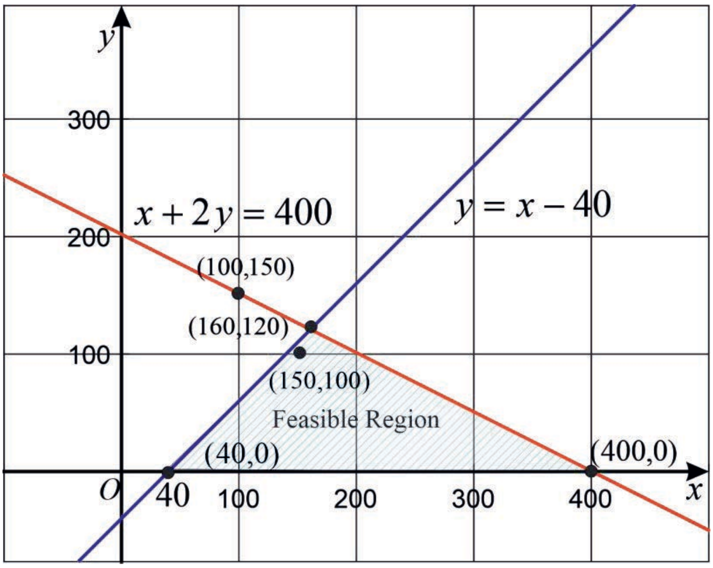


Figure 4.11

Example 3

Graph the solution set of the system

$$\begin{cases} y \leq -2x + 22 \\ y \leq -x + 13 \\ y \leq -\dfrac{2}{5}x + 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution

Draw the lines: $y = -2x + 22$, $y = -x + 13$ and $y = -\frac{2}{5}x + 10$. All the first three inequalities are $\leq$. Then, shade the region on the first quadrant below those lines. Finally, you get the shaded region as shown in Figure 4.12.

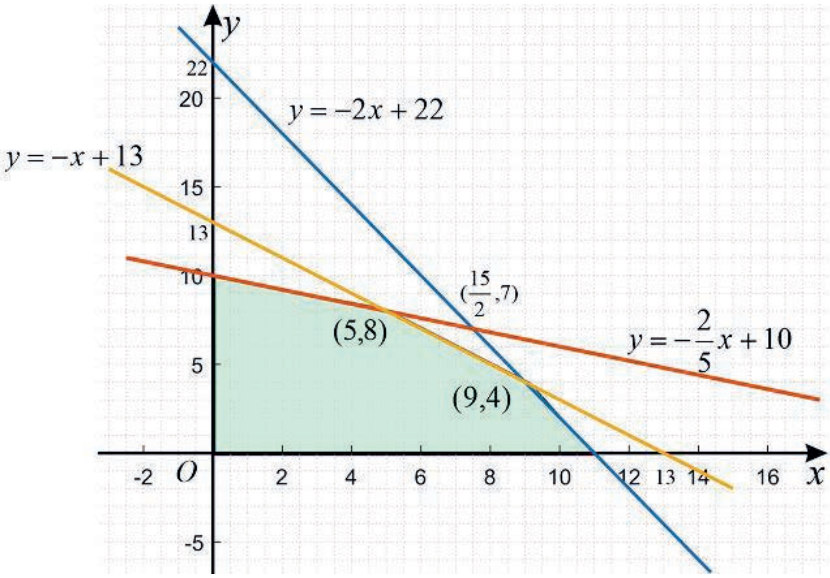


Figure 4.12

Exercise 4.5

1. Solve the system of linear inequalities $\begin{cases} y \leq -2x + 5 \\ x \geq 0 \\ y \geq 1 \end{cases}$

2. Solve the system $\begin{cases} y \leq -\frac{1}{2}x + 5 \\ y \leq -x + 8 \\ y \leq x \\ x \geq 0 \\ y \geq 0. \end{cases}$

3. A rectangular field with at most 200 meter of fencing is constructed. Write a linear inequality in terms of the length l and the width w. Sketch the graph of all possible solutions to this problem.

4. A company sells one product for Birr 8 and another for Birr 12. How many of each product must be sold so that revenues are at least Birr 2,400? Let x represent the number of products sold at Birr 8 and let y represent the number of products sold at Birr 12. Write a linear inequality in terms of x and y and sketch the graph of all possible solutions.

5. Yiseresh sells her photographs at a Gallery at a street fair. At the start of the day, she wants to have at least 25 photos to display at her Gallery. Each small photo she displays costs her Birr 40.00 and each large photo costs her Birr 80.00. She doesn't want to spend more than Birr 2000.00 on photos to be displayed.
   a. Write a system of inequalities to model this situation.
   b. Graph the system.
   c. Can she display 20 small and 10 large photos?
   d. Can she display 10 small and 30 large photos?

4.2 Maximum and Minimum Values

Activity 4.4

A student has 100m metallic wire. And he wants to enclose a land that he has by the wire. Which shape maximizes the area enclosed by the wire and which shape minimizes the area? Discuss the answer in class.

Terminologies

Optimization problem A problem which seeks to maximize or minimize a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an optimization problem. A special but a very important class of optimization problems is linear programming problem.

Linear programming is defined as a field of mathematics that deals with the problem of finding the maximum and minimum values of a given linear expression (called **objective function**), where the variables are subject to certain conditions (called **linear constraints)** expressed as linear inequalities. Linear programming problems are of much interest because of their wide applicability in industry, commerce, management science and so on.

Constraints mean the situation in which we cannot just take any x and y when looking for the x and y that optimise our objective function. If you think of the variables x and y as a point (x, y) in the xy −plane then you call the set of all points in the xy-plane that satisfy our constraints are the **feasible region**. Any point in the feasible region is called a **feasible point.**

Before we proceed further, we now formally define some terms we have used above and shall be using in the linear programming problems:

Objective function Linear function $z = ax + by$, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

Feasible solutions are points within and on the boundary of the feasible region that represent feasible solutions of the constraints.

Optimal (feasible) solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**

Example 1

A furniture dealer deals in only two items - tables and chairs. He has Birr 50,000 to invest and has storage space of at most 60 pieces. A table costs Birr 2500 and a chair Birr 500. He estimates that he can make a profit of Birr 300 from the sale of one table and Birr 100 from the sale of one chair. He wants to know how many tables and chairs he should buy from the available money so as to maximize his total profit, assuming that he can sell all the items which he buys.

Solution

Mathematical formulation of the problem

Let x be the number of tables and y be the number of chairs that the dealer buys. Clearly, x and y must be non-negative. The dealer has two limitations - investment capital and storage capacity. Suppose he decides to buy only tables but not chairs. So, he can buy 50000 ÷ 2500, i.e., 20 tables. His profit in this case will be Birr (300 × 20) = Birr 6,000.00.

On the other hand, let's suppose he chooses to buy only chairs but not tables. With his capital of Birr 50,000, he can buy 50000 ÷ 500, i.e., 100 chairs. But he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of (60 × 100) =Birr 6,000.00. There are many other possibilities. The dealer is constrained by the maximum amount he can invest and maximum number of items he can store.

$2500x + 500y \leq 50000$ or $5x + y \leq 100$, investment limitation (1)

$x + y \leq 60$, storage limitations (2)

$x \geq 0,\ y \geq 0$, non-negativity constraints (3)

We have to maximize the linear function z subject to certain conditions determined by a set of linear inequalities with variables as non-negative.

If the dealer wants to maximize profit, $Z = 300x + 100y$ is a linear objective function. Variables x and y are called decision variables.

Definition 4.3

Suppose f is a function with domain $I = \{x : a \leq x \leq b\}$

I. A number $M = f(c)$ for some c in I is called the **maximum** value of f on I, if $M \geq f(x)$, for all x in I.

II. A number $m = f(d)$ for some d *in* I is called the **minimum** value of f on I, if $m \leq f(x)$, for all x in I.

III. A value which is either a maximum or a minimum is called an **optimum** (or **extremum**) value of f on I.

You will now discuss how to find solutions to a linear programming problem. In this section, you will be concerned only with the graphical method.

Graphical method of solving linear programming problems

In Section 4.1, you learnt how to graph a system of linear inequalities involving two variables x and y and to find its solutions graphically.

Let us refer to the problem of investment in tables and chairs discussed in Example 1. You will now solve this problem graphically. Let us graph the constraints stated as linear inequalities (1)- (3). The graph of this system (shaded region) consists of the point's common to all half planes determined by the inequalities (1) to (3). Each point in this region represents a **feasible choice** open to the dealer for investing in tables and chairs.

In Figure 4.13(a), the region *OABC* is the feasible region for the problem. The region other than feasible region is called an **infeasible region**.

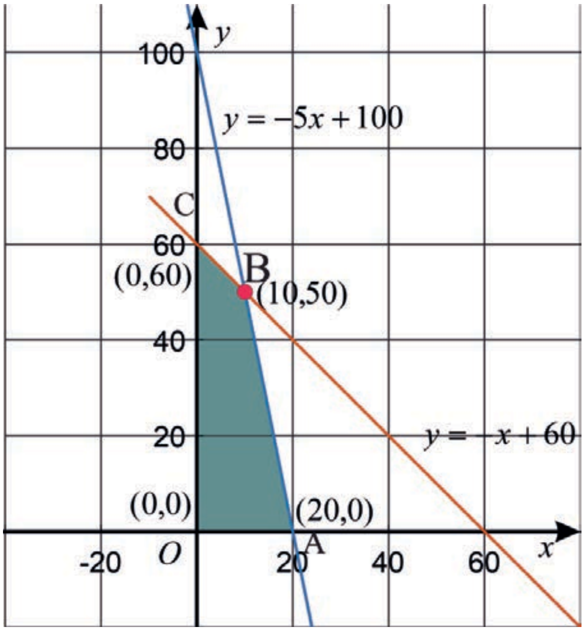


Figure 4.13(a)

In Figure 4.13(a), every point within and on the boundary of the feasible region *OABC* represents feasible solution to the problem. For example, the point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0) etc. The point (25, 40) is an infeasible solution of the problem.

You see that every point in the feasible region *OABC* satisfies all the constraints given in (1) to (3). Since there are **infinitely many points**, it is not evident how we should go about finding a point that gives a maximum value of the objective function $Z = 300x + 100y$.

The possible candidate to give a maximum value are the corner points (vertices) of the bounded (feasible) region. Such points are: *O, A, B* and *C* and it is easy to find their coordinates as (0, 0), (20, 0), (10, 50) and (0, 60) respectively. Let us now compute the values of Z at these points. The Z values will be obtained by substituting the coordinates of the corners into the objective function

$$Z = 300x + 100y,$$

At O, $Z(0,0) = 300 \times 0 + 100 \times 0 = 0.$

At A, $Z(20,0) = 300 \times 20 + 100 \times 0 = 6000.$

At B, $Z(10,50) = 300 \times 10 + 100 \times 50 = 3000 + 5000 = 8000.$

At C, $Z(0,60) = 300 \times 0 + 100 \times 60 = 0 + 6000 = 6000.$

Accordingly, you observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs. This method of solving linear programming problem is referred as **Corner Point Method**. You can verify the solution as follows. Solving the objective function for y,

$$Z = 300x + 100y$$

$$y = -3x + \frac{Z}{100}$$

This means, it is a line, whose slope is -3, and *y*-interception is $\frac{Z}{100}$. We shall move this line within the region to find the maximum value of y-interception, and the value of Z.

As shown in the Figure 4.13(b), the value of *y*-interception will be maximum, when the line passes through B (10, 50). Then, the value of *Z* will be:

$$y = -3x + \frac{Z}{100}$$

$$50 = -3 \times 10 + \frac{Z}{100}$$

$$80 = \frac{Z}{100} \Leftrightarrow Z = 8000$$

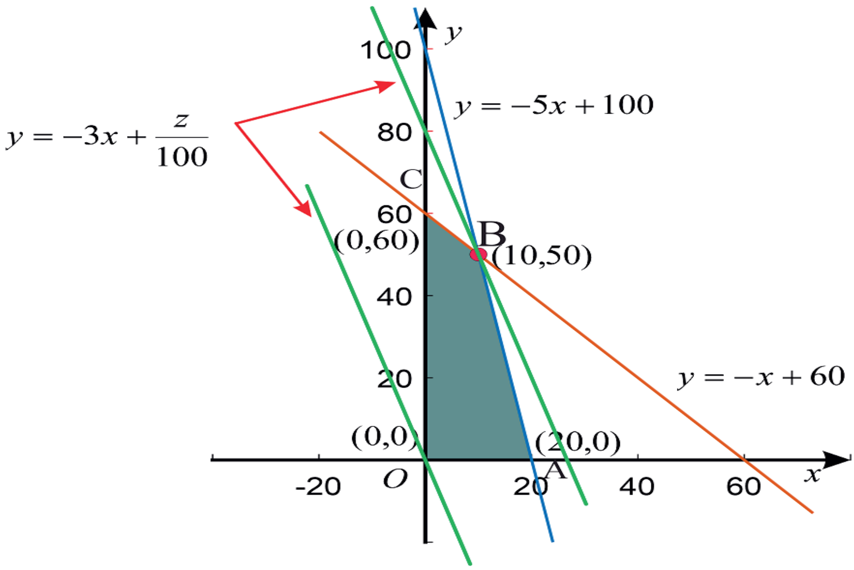


Figure 4.13(b)

Exercise 4.6

1. A furniture dealer deals in only two items - tables and chairs. He has Birr 50,000 to invest and has storage space of at most 60 pieces. A table costs Birr 2500 and a chair Birr 500. Suppose that the dealer can make profit of Birr 200 from the sale of one table and Birr 50 from that of a chair.
   a. Find the linear objective function of the profit.
   b. What is the maximum profit can the dealer make?
2. Consider an ice-cream manufacturing company that produces only two types of ice-cream A and B. Both the ice-cream require Milk and Cream only. To manufacture each unit of A and B, the following quantities are required: Each unit of A requires 1 unit of Milk and 3 units of Cream. Each unit of B requires 1 unit of Milk and 2 units of Cream. The company kitchen has a total of 5 units of Milk and 12 units of Cream. On each sale, the company makes a profit of: Birr 6 per unit A sold, Birr 5 per unit B sold. Now, the company wishes to maximize its profit. How many units of A and B should it produce respectively?

4.2.1 Steps to Solve a Linear Programming Problem

Definition 4.4

A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

Definition 4.5

A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a line. Otherwise, it is called unbounded. Unbounded means that the feasible region extends indefinitely in any direction.

We use the following theorems which are fundamental in solving linear programming problems. However, the proofs of these theorems are beyond the scope of the textbook.

Theorem 4.1

Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 4.2

Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is **bounded**, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.

Remark

If R is **unbounded**, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (By Theorem 4.1).

Steps to solve a linear programming problem by corner point method

Step 1: Draw the graph of the feasible region.

Step 2: Determine the coordinates of the corner points either by inspection or by solving the two equations of the lines intersecting at that point.

Step 3: Evaluate the objective function at each corner points. Let M and m be the largest and smallest values of these points respectively.

Step 4: If the feasible region is bounded, M and m are the maximum and minimum values of Z. If the feasible region is unbounded:

i. M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

ii. Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Optimal solutions always exist when the feasible region is bounded, but may or may not exist when it is unbounded. You will now illustrate these steps of Corner Point Method by considering additional examples:

Example 1

Solve the linear programming problem by Corner Point Method.

Maximize $Z = x + y$, subject to the following constraints:

$$\begin{cases} x + 2y \leq 4 \\ x - y \leq 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution

The given constrains are changed to

$$\begin{cases} y \leq -\frac{1}{2}x + 2 \\ y \geq x - 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Draw the region of constrains by drawing the lines of equations:

$$\begin{cases} y = -\frac{1}{2}x + 2 \\ y = x - 1 \\ x = 0 \\ y = 0 \end{cases}$$

As shown in the Figure 4.14, the region is bounded. The corner points of the region are (0, 0), (1, 0), (0, 2) and (2, 1). We can evaluate Z at each point as follows:

$$Z = x + y$$

When (0, 0), then $Z = 0 + 0 = 0$

When (1, 0), then $Z = 1 + 0 = 1$

When (0, 2), then $Z = 0 + 2 = 2$

When (2, 1), then $Z = 2 + 1 = 3$

Or solve the objective function for y, $y = -x + Z$. The value Z is the y-intercept of the line. When the line passes the point (2, 1), the value Z has the maximum value of 3. Therefore, the maximum value of Z is 3, and it occurs when $x = 2, y = 1$.

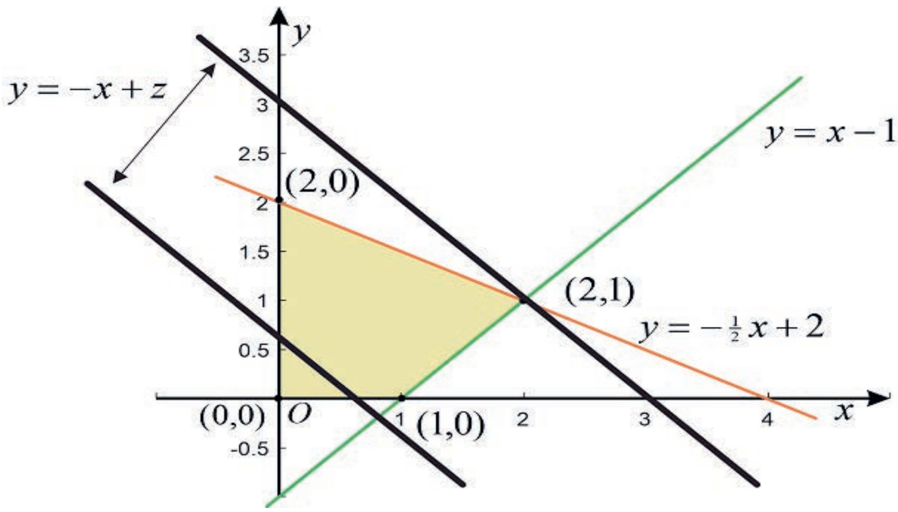


Figure 4.14

Example 2

Solve the linear programming problem by Corner Point Method.

Maximize $Z = 3x + 2y$, subject to the same constraints as Example 1:

Solution

From the constraints you sketched the feasible region shown in Figure 4.14 and observed that the feasible region is **bounded.** So, you can test each vertex to determine the maximum value of *Z*. The corner points of this region are (0, 0), (1, 0), (2, 1) and (0, 2) and you can evaluate *Z* at each corner point.

Corner points (vertices)	Corresponding Values of Z
(0, 0)	$3 \times 0 + 2 \times 0 = 0$
(1, 0)	$3 \times 1 + 2 \times 0 = 3$
(2, 1)	$3 \times 2 + 2 \times 1 = 8$
(0, 2)	$3 \times 0 + 2 \times 2 = 4$

Thus, the maximum value of *Z* is 8, and occurs when $x = 2$ and $y = 1$.

Exercise 4.7

Solve the following Linear Programming Problems by Corner Point Method:

a. Maximize $Z = x + y$,

$$\text{Subject to: } \begin{cases} 2x + 3y \leq 12 \\ 2x + y < 8 \\ x \geq 0, y \geq 0 \end{cases}$$

b. Find values of x and y that maximize,

$$Z = x + 3y,$$

$$\text{Subject to: } \begin{cases} 2x + 3y \leq 24 \\ x - y < 7 \\ y \leq 6 \\ x \geq 0, y \geq 0 \end{cases}$$

c. Minimize: $Z = -40x + 20y$,

$$\text{Subject to: } \begin{cases} 2x - y \geq -5 \\ 3x + y \geq 3 \\ 2x - 3y \leq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

Further on a Linear Programming Problem

Activity 4.5

Draw the region enclosed by the following lines and identify the vertices.

a. $2x + 5y \leq 10, x \geq 0, y \geq 0.$

b $x + y \leq 40,\ 2x + 3y \leq 72, x \geq 0, y \geq 0.$

Example 1

Find the maximum value of $Z = 4x + 6y$

$$\text{Subject to } \begin{cases} y \le x+1 \\ y \le -x+27 \\ y \le -\dfrac{2}{5}x+18 \\ x \ge 0, y \ge 0 \end{cases}$$

Solution

The region is determined by drawing the lines,

$$\begin{cases} y = x+1 \\ y = -x+27 \\ y = -\frac{-2}{5}x+18 \end{cases}$$

The resulting bounded region coordinates are $(0,0), (27,0), (15,12)$, $(\frac{85}{7}, \frac{92}{7})$ and $(0,1)$. Solving the objective function for y, $y = -\frac{2}{3}x + \frac{Z}{6}$.

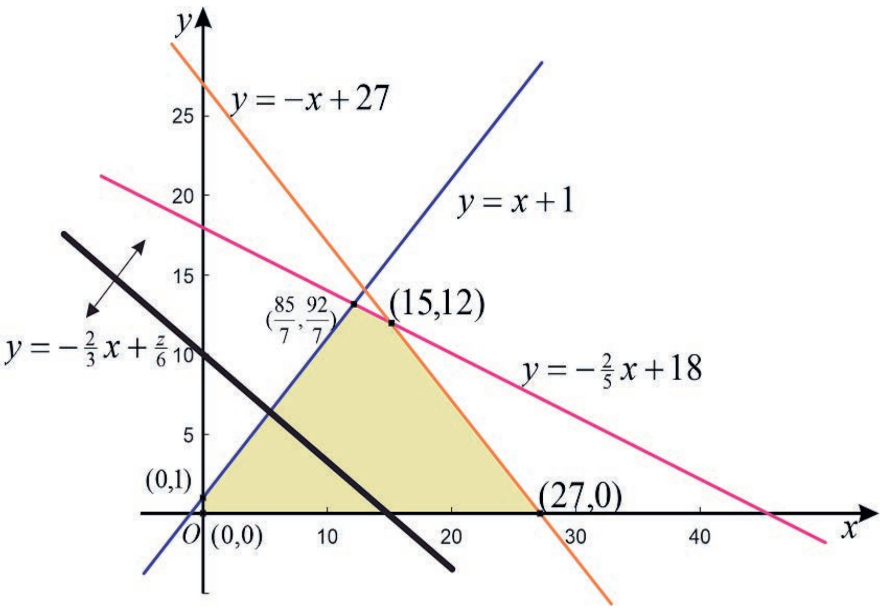


Figure 4.15

Testing the objective function at the corner points gives the following.

Corner points	Value of $Z = 4x + 6y$
(0, 0)	0
(27, 0)	108
(15, 12)	132
$(\frac{85}{7}, \frac{92}{7})$	892/7 or 127.43
(0, 1)	6

Thus, the maximum value of Z is 132 when $x = 15$ and $y = 12$.

Example 2

Find values of x and y which minimize the value of the objective function, $Z = 2x + 4y$,

Subject to $\begin{cases} y \geq -\frac{1}{2}x + 5 \\ y \geq -3x + 10 \\ x \geq 0, y \geq 0 \end{cases}$

Solution

From the given constraints the feasible region is as shown in Figure 4.16, which is unbounded. So the linear programing problem doesn't have maximum value. The corner points are at (0, 10), (2, 4) and (10, 0) with values given below.

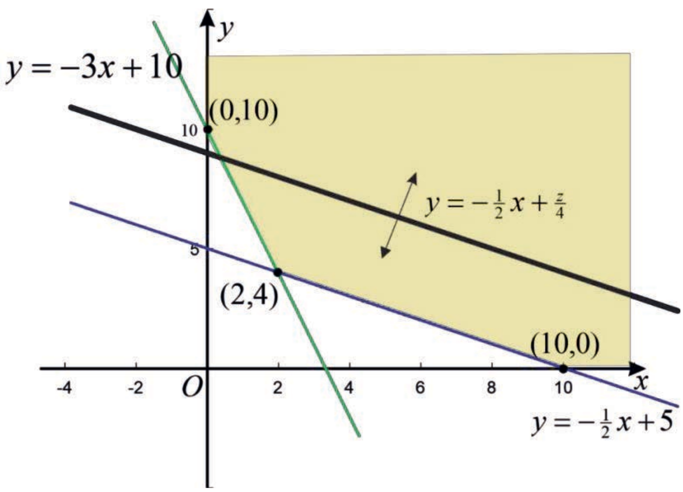


Figure 4.16

Here points (2, 4) and (10, 0) give the minimum value $Z = 20$. So, the solution is not unique. In fact, every point on the line segment through (2, 4) and (10, 0) gives the same minimum value of $Z = 20$.

Corner points	Values of Z
(0, 10)	$2\times0+4\times10=40$
(2, 4)	$2\times2+4\times4=20$
(10, 0)	$2\times10+4\times0=20$

From the above example, we can observe that

i. An optimization problem can have infinite solutions.

ii. Not all optimization problems have a solution, for instance, the problem in Example 2 doesn't have a unique set of (x, y) value corresponding to the minimum value for Z.

Exercise 4.8

Solve the following Linear Programming Problems graphically:

a. Maximize $Z = 3x + 4y$

Subject to the constraints: $\begin{cases} x + y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$

b. Maximize or Minimize $Z = 5x + 6y$

Subject to $\begin{cases} 2x + y \leq 16 \\ x - y \leq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$

c. Maximize or Minimize $Z = 2x + 3y$

Subject to $\begin{cases} 3x + 7y \leq 42 \\ x + 5y \leq 42 \\ x \geq 0 \\ y \geq 0 \end{cases}$

4.3 Applications

In this section, you shall illustrate the application of linear programming to different real world problems. Some of the known linear programming problems are listed below:

4.3.1 Real Life Problems

A. Diet Problems: The basic idea behind the diet problem is choosing quantities of foods so as to satisfy nutritional requirements and ensure that the price of the resulting diet is within a reasonable range. Two versions of the problem can, at least in theory, be thought of: either we minimize the cost of the diet and ensure by way of the constraints that some nutritional constraints are satisfied, or we maximize the nutritional content of the diet subject to a budget constraint.

Example 1

World Food Program (WFP) uses at least 800 kg of special feed daily in conflict areas. A special feed is a mixture of corn and soybean meal with the composition of a kilogram of corn that contains 10% protein and 2% fiber costs Birr 30 per kg. A kilogram of soybean meal constitutes 60% protein and 6% fiber and costs Birr 90 per kg. The dietary requirements for the special feed are at least 30% protein and at most 4% fiber. WFP wishes the daily minimum cost feed mix.

Solution

The first step in solving such real life Linear Programming problems is to define the relevant decision variable. In any linear programming model, the decision variables should completely describe the decisions to be made. We define the two decision variables:

x is the number of kilogram of corn in the daily mix,

y is the number of kilogram of soybean meal in the daily mix.

The information is summarized as

	Content		Cost (Birr/kg)	Mixture in a day (kg)
	Protein (%)	Fiber (%)		
Corn	10	2	30	x
Soybean	60	6	90	y

In any linear programming problem (LPP), the decision maker wants to maximize (usually revenue or profit) or minimize (usually costs) which are some function of the decision variables. Formulation of the objective function becomes the second step. The objective function that seeks to minimize the total daily cost of the feed mix is expressed as

Minimize $Z = 30x + 90y$

The 3rd step is to construct the constraint (or restriction) on decision variables. In this example, the constraints reflect the daily amount needed and the dietary requirements.

Because WFP needs at least 800 kg of feed a day, the associated constraint is expressed as: $x + y \geq 800.$

As for the protein daily requirement constraints, the amount of protein included in x kg of corn and y kg of soybean meal is $0.1x + 0.6y$ kg. This quantity should equal at least 30% of the total feed mix; i.e. $0.1x + 0.6y \geq 0.3(x + y)$.

In a similar manner, the fiber requirement at most 4% is constructed as;

$0.02x + 0.06y \leq 0.04(x + y) = 0.04x + 0.04y.$

The complete LPP model becomes

Minimize $Z = 30x + 90y$

Subject to: $\begin{cases} x + y \geq 800 \\ 0.2x - 0.3y \leq 0 \\ 0.02x - 0.02y \geq 0 \\ x \geq 0, y \geq 0 \end{cases}$ equivalently, $\begin{cases} y \geq -x + 800 \\ y \geq \dfrac{2}{3}x \\ y \leq x \\ x \geq 0, y \geq 0 \end{cases}$

The above three steps are collectively known as mathematical formulation or mathematical modelling of a real world.

To solve the problem graphically, you draw the lines $y = -x + 800$, $y = \frac{2}{3}x$, and $y = x$ in the same coordinate axis. Shade the region above the lines $y = -x + 800$, $y = \frac{2}{3}x$, and below the line $y = x$.

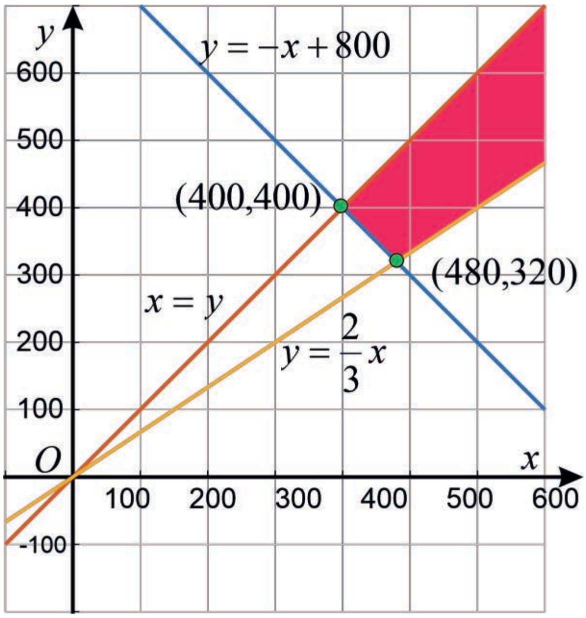


Figure 4.17

From the Figure 4.17, the possible optimum solution can be obtained at the points $(400, 400)$ and $(480, 320)$. These points are the intersections of the lines $y = -x + 800$ and $y = \frac{2}{3}x$, and $y = x$.

Testing the points at $(400, 400)$.

$$Z = 30 \times 400 + 90 \times 400 = 48{,}000$$

At $(480, 320)$, $Z = 30 \times 480 + 90 \times 320 = 43{,}200$. The associated minimum cost of the feed mix is Z= Birr 43,200 per day.

Exercise 4.9

1. Assume that there are two products, cereal and milk, for breakfast and assume that a person must consume at least 60 units of iron and 70 units of protein to stay alive. Assume that one unit of cereal costs Birr 120 and contains 30 units of iron and 5 units of protein and one unit of milk costs Birr 30 and contains 15 units of iron and 10 units of protein. Find the cheapest diet which will satisfy the minimum daily requirement.
2. A dietician has to develop a special diet using two foods M and N. Each packet of food M contains 15 units of calcium, 6 units of iron, 7 units of cholesterol and 9 units of vitamin A. Each packet of the same quantity of food N contains 5 units of calcium, 22 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires at least 245 units of calcium, at least 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimize the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

B. Manufacturing Problems: In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labor hour per unit of product, warehouse space per unit of the output, etc. in order to make maximum profit.

Example 1

A furniture company makes two models A and B of office furniture. Each piece of Model A requires 9 labor hours for fabricating and 1 labor hour for finishing. Each piece of Model B requires 12 labor hours for fabricating and 3 labor hours for finishing. For fabricating and finishing, the maximum labor hours available are 180 and 30 respectively. The company makes a profit of Birr 8,000 on each piece of model A and Birr 12,000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realize a maximum profit? What is the maximum profit per week?

Solution

First construct the mathematical expression. The decision variables are x as the number of pieces of Model A and y as the number of pieces of Model B both manufactured by the company. The information is summarized in the following table.

Model	Required labor force (hour)		Profit (Birr/piece)	Piece per week
	Fabricating	Finishing		
A	9	1	8000	x
B	12	3	12000	y

Then total profit in Birr is $Z = 8000x + 12000y$

Thus, you have the following mathematical model for the given problem.

Maximize $Z = 8000\,x + 12000\,y$ (1)

Subject to the constraints:

$9x + 12y \leq 180$ (Fabricating constraint) (2)

$x + 3y \leq 30$ (Finishing constraint) (3)

$x \geq 0, y \geq 0$ (Non-negative constraints) (4)

The feasible region (the shaded) determined by the linear inequalities (2)-(4) is shown in the Figure 4.18. Note that the feasible region is bounded.

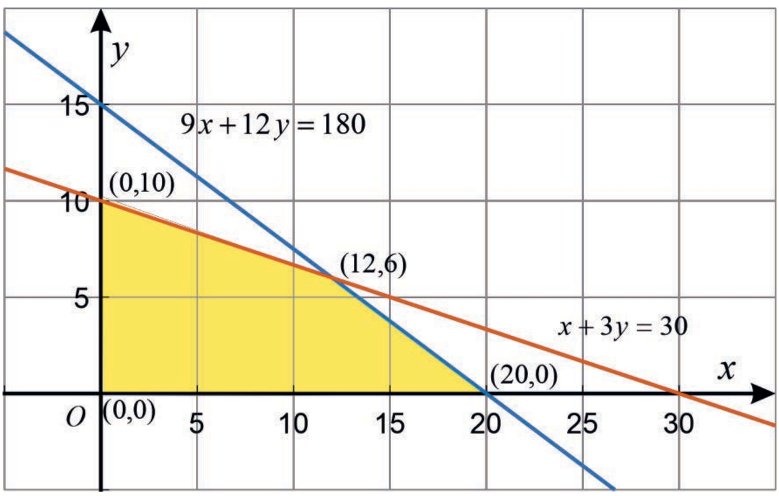


Figure 4.18

Corner point	$Z = 8000x + 12000y$
(0, 0)	0
(20, 0)	160000
(12, 6)	168000
(0, 10)	120000

The objective function (1) at each corner point is summarized in the table. From this, you find that maximum value of Z is 168,000 at B (12, 6). Hence, the company should produce 12 pieces of Model A and 6 pieces of Model B to realize a maximum profit Birr 168,000.

Example 2

XY car manufacturer produces two basic models, the M13 and M14. These cars are sold to dealers at a profit of Birr 20,000 per M13 and Birr 10,000 per M14. A M13 requires 150 hours for assembly, 50 hours for painting and finishing and 10 hours for checking and testing. The M14 requires 60 hours for assembly, 40 hours for painting and finishing and 20 hours for checking and testing. The total number of hours available per month is: 30,000 in the assembly department, 13,000 in the painting and finishing department and 5,000 in the checking and testing department. Let x be the number of M13 and y be the number of M14 models manufactured per month. From this:

a. Write down the set of constraint inequalities.

b. Use graph paper to represent the set of constraint inequalities.

c. Shade the feasible region on the graph paper.

d. Write down the profit generated in terms of x and y.

e. How many cars of each model must be produced in order to maximize the monthly profit?

f. What is the maximum monthly profit?

Solution

The information in the problem is summarized as follows:

Department	Model 13	Model 14	Hours available per month
Assembly	150	60	30,000
Painting and finishing	50	40	13,000
Checking and testing	10	20	5,000

a. Adding number of hours required for assembling each model,

$$150x + 60y \leq 30000.$$

Adding number of hours required for painting and finishing each model,

$$50x + 40y \leq 13000.$$

Adding number of hours required for checking and testing each model,

$$10x + 20y \leq 5000.$$

b. Drawing the constraints in coordinate plane we obtain the graph in Figure 4.19.

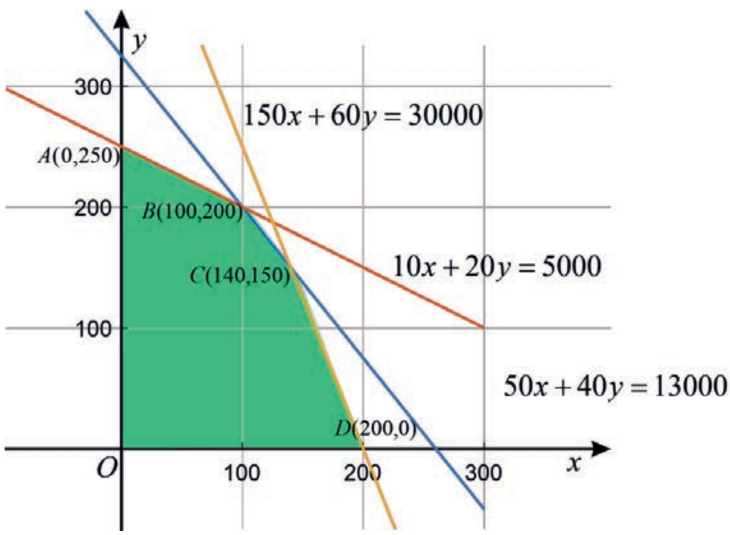


Figure 4.19

The region which satisfies all the three constraints are enclosed with vertices $O\ (0, 0), A\ (0, 250), B\ (100, 200), C\ (140, 150),$ and $D\ (200, 0)$.

c. The feasible region is the shaded region with corner points OABCD.

d. The company gets a profit for a unit sell of M13 Birr 20,000 and a unit sell of M14 is Birr 10,000. This is summarized as $Profit = 20{,}000x + 10{,}000y$; the total profit the company earns per month.

e. To obtain a product mix which will give the maximum profit, we test each corner point of the shaded region.

Corner points	$Profit = 20000x + 10000y$
O (0, 0)	0
A (0, 250)	2,500,000
B (100, 200)	4,000,000
C (140, 150)	4,300,000
D (200, 0)	4,000,000

f. To maximize the objective function, we select Point C. The maximum profit can be made if 140 Model M13 cars and 150 Model M14 cars are sold. The maximum possible profit is Birr 4,300, 000.

Exercise 4.10

1. Assume that in the context of a production problem there are two products, P1 and P2, which are manufactured on three machines, M1, M2, and M3. Each product has to be processed on each of the three machines, but the order in which the products are processed on the machines is assumed to be immaterial. The two products are sold for the (fixed) unit price of Birr 12 and Birr 27, respectively, while the capacities of the three machines are 15, 14, and 12 hours per day, respectively. Assume now that it takes 9 minutes to process one unit of P1 on M1, 6 minutes to process one unit of P1 on M2, and 10 minutes to process one unit of P1 on M3. The corresponding data for P2 are 12, 14, and 18. Finally, it is required that at least 70 product units are made and sold. Formulate the production problem and determine the optimal solution.
2. A wood products firm uses left over time at the end of each week to make goods for stock. Currently, there are two products on the list of items that are produced

for stock; a chopping board and a knife holder. Both items require three operations: cutting, gluing and finishing. The manager of the firm has collected the following data on these products:

Item	Profit per unit in Birr	Time per unit (minutes)		
		Cutting	Gluing	Finishing
Chopping board	20	1.4	5	12
Knife holder	60	0.8	13	3

The manager has also determined that during each week 56 minutes of cutting, 650 minutes of gluing and 360 minutes of finishing time are available.

Required:

a) Formulate the problem as a linear programming model.

b) Determine to maximize the profit of each decision variables.

3. A company is making two products A and B. The cost of producing one unit of product A and B is Birr 60 and Birr 80 respectively. As per the agreement, the company has to supply at least 200 units of product B to its regular customers per month. One unit of product A requires one machine hours whereas product B has machine hours available abundantly within the company. Total machine hours available for product A are 400 hours. One unit of each product A and B requires one labor hour each and total of 500 labor hours are available. The company wants to minimize the cost of production by satisfying the given requirement. Formulate the problem as a linear programming problem and find the minimum cost.

C. Allocation Models: the most practical applications of linear programming problem is showed in allocation models. All allocation models in common attempt to allocate a scarce resource so as to optimize the consequence of that allocation. These allocations are to be performed so as to maximize their profit, minimize their cost, or optimize other efficiency criteria specified by the decision maker.

Example 1

A patient in a hospital is required to have at least 84 units of drug A and 120 units of drug B each day. Each gram of substance M contains 10 units of drug A and 8 units of drug B, and each gram of substance N contains 2 units of drug A and 4 units of drug B. Suppose both substances M and N contain an undesirable drug C, 3 units per gram in M and 1 unit per gram in N. How many grams of each substance M and N should be mixed to meet the minimum daily requirements and at the same time minimize the intake of drug C? How many units of drug C will be in this mixture?

Solution

Define the decision variable as x number of grams of substance M used, y number of grams of substance N used.

	Content (unit)		Amount of undesirable drug C (unit)	Amount (g)
	A	B		
M	10	8	3	x
N	2	4	1	y

The objective of this allocation problem is to minimize drug C from substances M and N. The constraints are the minimum requirements of each drug

$10x + 2y \geq 84$ . . . . from drug A

$8x + 4y \geq 120$ . . . . from drug B.

Combining the above, we obtain the LPP.

$$\text{Minimize } C = 3x + y$$

$$\text{Subject to: } \begin{cases} 10x + 2y \geq 84 \\ 8x + 4y \geq 120 \\ x \geq 0, y \geq 0 \end{cases}$$

The feasible region can be determined by drawing the lines $y = -5x + 42$ and $y = -2x + 30$ both inequalities are greater than or equal to ($\geq$), you shade the region above the lines. Finally you get the region painted by green color in Figure 4.20.

The proposed optimal points are (0, 42), (4, 22) and (15, 0). The values of the objective function at these points are, C at (0, 42) is 42, C at (4, 22) is 34 and C at (15, 0) is 45. The minimum intake of drug C is 34 units and it attained by taking of 4 grams of substance M and 22 grams of substance N.

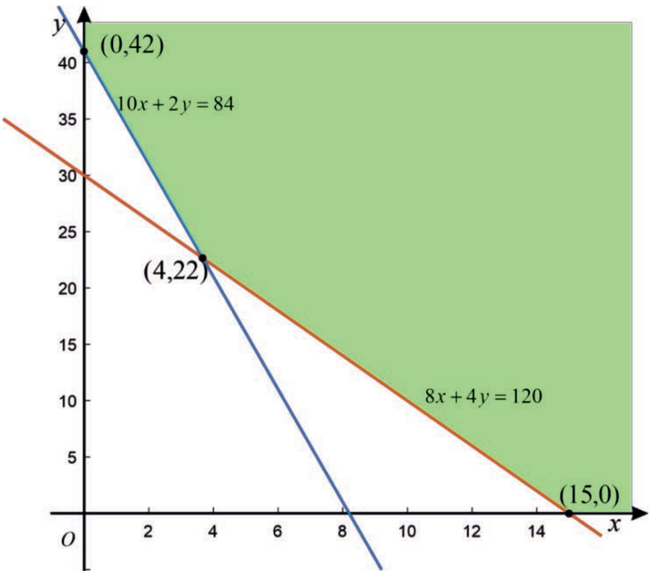


Figure 4.20

Exercise 4.11

1. One of the Ethiopian Airline airplanes can carry a maximum of 320 passengers. A profit of Birr 750 is made on each business class ticket and a profit of Birr 450 is made on each economy class ticket. The airline reserves at least 20 seats for business class. However, at least 4 times as many passengers prefer to travel by economy class than by the business class. Determine how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit?

D. Transportation Problems: In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Example 1

The officers of a high school senior class are planning to rent buses and vans for a class trip. Each bus can transport 45 students, requires 4 supervisors and costs Birr 4,000 to rent. Each van can transport 12 students, requires 1 supervisor, and costs Birr 600 to rent. The officers must plan to accommodate at least 420 students. Since only 48 parents have volunteered to serve as supervisors, the officers must plan to use at most 48 supervisors. How many vehicles of each type should the officers rent in order to minimize the transportation costs? What is the minimum transportation cost?

Solution

To solve this application problem first we define the decision variables as x is the number of buses to rent, y is the number of vans to rent.

The objective function is the cost function, $Z = 4000x + 600y$

These are two limitations, number of students to be accommodated is at least 420 and supervisors are at most 48, and this is expressed as

$4x + y \leq 48,$ Supervisor's limitation

$45x + 12y \geq 420,$ Student's accommodation

$x \geq 0, y \geq 0.$ Non negativity

The LPP is

Minimize $Z = 4000x + 600y$ (1)

Subject to $\begin{cases} y \leq -4x + 48 \\ y \geq \dfrac{-15}{4}x + 35 \\ x \geq 0, y \geq 0 \end{cases}$

The feasible region is determined by drawing the lines $y = -4x + 48$ and $y = -\frac{15}{4}x + 35$. By shading the regions below the first line and above the second we get the Figure 4.21.

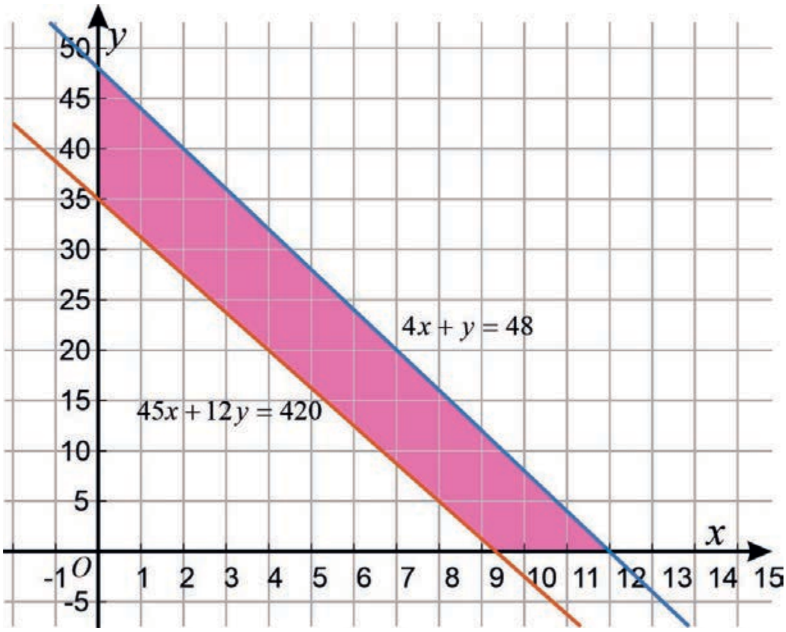


Figure 4.21

The proposed optimal points are (0, 48), (0, 35), (12, 0) and $(\frac{28}{3}, 0)$. The values of the objective function at these points are:

At (0, 48), $Z = 600 \cdot 48 = 28{,}800$.

At (0, 35), $Z = 600 \cdot 35 = 21{,}000$.

At (12, 0), $Z = 4000 \cdot 12 = 48{,}000$.

At $(\frac{28}{3}, 0)$, $Z = 4000 \cdot \frac{28}{3} = 37{,}333.33 \ldots \approx 37{,}333$. Thus, the minimum cost is 21,000 Birr.

Example 2

There are two factories one is located at industrial Park 1 and the other at industrial Park 2. From these parks, a certain commodity is to be delivered to each of the three markets situated at A, B and C. The weekly requirements of the markets are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at Park 1 and Park 2 are respectively 8 and 6 units. The cost of transportation per unit is given in the table below:

From/To	Cost in Birr		
	A	B	C
Park 1	160	100	150
Park 2	100	120	100

How many units should be transported from each factory to each market in order that the transportation cost is minimum? What will be the minimum transportation cost?

Solution

The problem can be explained diagrammatically in Figure 4.22.

Let x units and y units of the commodity be transported from the factory at Park 1 to the markets at A and B respectively. Then $(8 - x - y)$ units will be transported to market at C. From these we have $x \geq 0, y \geq 0$, and $8 - x - y \geq 0$ equivalent to $x + y \leq 8$.

Now, the weekly requirement of the market at A is 5 units of the commodity. Since x units are transported from the factory at Park 1, the remaining $(5 - x)$ units need to be transported from the factory at Park 2. Obviously, $5 - x \geq 0, i.e. x \leq 5$.

Similarly, $(5 - y)$ units are to be transported from the Park 2 to the market at B and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the Park 2 to the market at C. Thus, $5 - y \geq 0, x + y - 4 \geq 0, i.e. y \leq 5, x + y \geq 4$.

Total transportation cost Z is given by

$$Z = 160x + 100y + 150(8 - x - y) + 100(5 - x) + 120(5 - y) + 100(x + y - 4)$$

$$= 10x - 70y + 1900$$

$$= 10(x - 7y + 190)$$

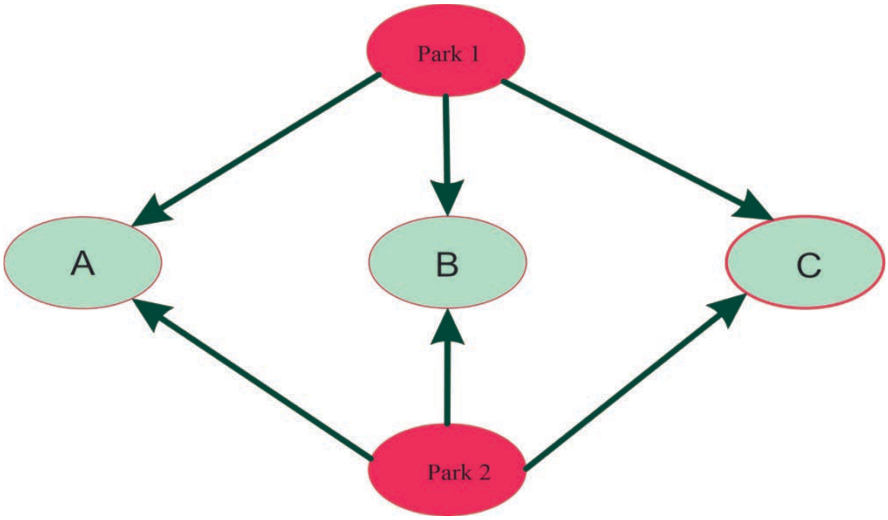


Figure 4.22

Therefore, the problem is expressed as: Minimize $Z = 10(x - 7y + 190)$.

Subject to the constraints: $\begin{cases} x + y \leq 8 \\ x \leq 5,\ y \leq 5 \\ x + y \geq 4 \\ x \geq 0, y \geq 0 \end{cases}$

Drawing the constraint inequality in the xy –plane you obtain the feasible region enclosed by *ABCDEF*. Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0).

Let us evaluate Z at these points.

Corner points	$Z = 10x - 70y + 1900$
(0, 4)	1620
(0, 5)	1550
(3, 5)	1580
(5, 3)	1740
(5, 0)	1950
(4, 0)	1940

From the table, we see that the minimum value of Z is 1550 at the point (0, 5). Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at Park 1 and 5, 0 and 1 units from the factory at Park 2 to the market at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Birr 1550.

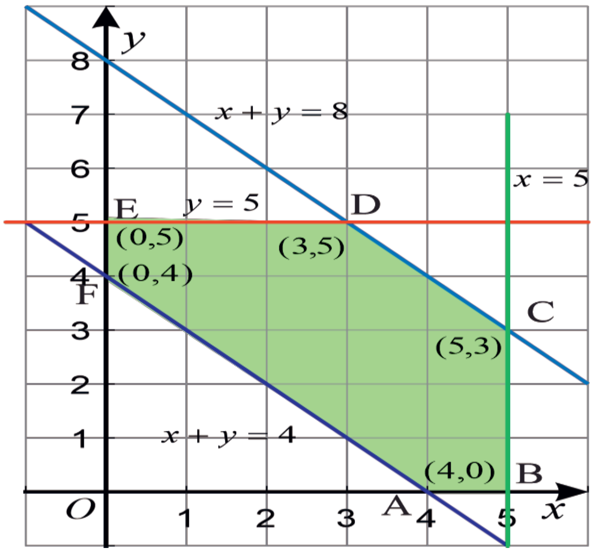


Figure 4.23

Exercise 4.12

1. An oil company has two depots A and B with capacities of 7000L and 4000L respectively. The company is to supply oil to three petrol stations, D, E and F whose requirements are 4500L, 3000L and 3500L respectively in Addis Ababa. The distances (in km) between the depots and the petrol stations are given in the following table. Assuming that the transportation cost of 10 liters of oil is Birr 2 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

Distance in kilometer		
From/To	A	B
D	7	3
E	6	4
F	3	2

4.3.2 Solving Linear Programming Problems Using Spreadsheet

In the previous sections, you discussed how to solve LPP graphically. Now, you will see how to solve LPP by the help of computer applications in particular spreadsheet.

Activity 4.6

Discuss the main features of Microsoft Excel in a group.

A mathematical model implemented in a spreadsheet is called a spreadsheet model. Major spreadsheet packages come with a built-in optimization tool called Solver. Now we demonstrate how to use Excel spreadsheet modeling and Solver to find the optimal solution of optimization problems.

If the model has two variables, the graphical method can be used to solve the problem. Very few real world problems involve only two variables. For problems with more than two variables, we need to use complex techniques and tedious calculations to find the optimal solution. The spreadsheet solver approach makes solving optimization problems a fairly simple task.

Nowadays, there are various computer applications available for solving LP problems. MS Excel useful to solve LP problems with its "SOLVER" application. Hence, the Solver application should be installed and appear in the Excel toolbar.

To install solver in Excel, click **"File"** and select **"Excel Options",** where click on **"Add-Ins".** You can find **"Manage: Excel Add-Ins"** at the bottom of the browser. Then, click on **"Go",** where you find an Add-Ins message box and select the option ***"Solver Add-In"*** and finally click ***"OK".*** Now, you can find the icon ***"Solver"*** in ***"Analysis"*** the subsection of Data toolbar in the main menu to solve your LP problems.

The first step is to organize the spreadsheet to represent the model. We use separate cells to represent decision variables, create a formula in a cell to represent the objective function and create a formula in a cell for each constraint left hand side.

Once the model is implemented in a spreadsheet, next step is to use the **Solver** to find the solution. In the Solver, we need to identify the locations (cells) of objective function, decision variables, nature of the objective function (maximize/minimize) and constraints.

Example 1

A Textile PLC produces bedsheets of types *S* and *L*. The wholesale price is Birr 360 per bedsheet for *S* and Birr 800 for *L*. Two time constraints result from the use of two machines M_1 and M_2. On M_1 needs 2 min for an *S* bedsheet and 8 min for an *L* bedsheet. On M_2 needs 5 min for an *S* bedsheet and 2 min for an *L* bedsheet. Determine production figures *x* and *y* for *S* and *L*, respectively; i.e., number of bedsheets produced per hour, so that the hourly revenue is maximized.

Solving LP problem in excel

The objective function and the constraint for the production problem is

Maximize $Z = 360x + 800y$

$2x + 8y \leq 60$ on machine M_1,

$5x + 2y \leq 60$ on machine M_2,

$x \geq 0, y \geq 0.$

Step 1: Put the problem in Excel. Put the objective function coefficients into a row with **at least 2 blank rows above it** with the constraint coefficients below. Label the rows down the left hand side in column 1. Leave one blank column after the last variable and label it **sum.**

Then put in the right hand side (RHS). Put names for each variable above the variables in the row just above the objective function coefficients. Label that row **Names of decision variables**. The resultant spreadsheet is

Step 1					
Name of Decision Variable	x	y	Sum	RHS	
Objective to maximize	360	800			
Available time on M1	2	8		60	
Available time on M2	5	2		60	

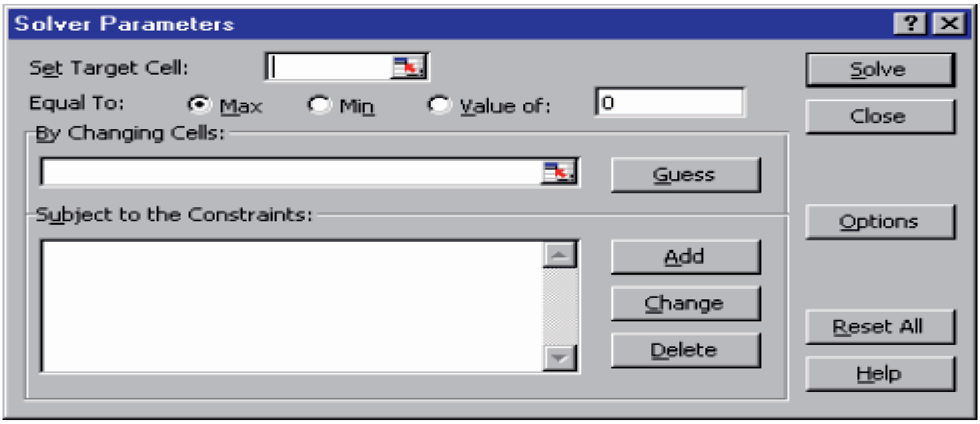
Step 2: Now, label the row just above tableau **Variable values to manipulate**. **Enter 0 values above the variables**. These are the cells that Excel will "change" to find the optimum solution to the problem.

Variable values to manipulate	0	0		
Name of Decision Variable	x	y	Sum	RHS
Objective to maximize	360	800		
Available time on M1	2	8		60
Available time on M2	5	2		60

Step 3: Now, construct Excel cell entries to add up each LP model equations. Place these in the column named Sum. These will involve adding the numbers in each equation times the numbers from the Variable values to manipulate row. Namely, with Variable values in row 8 and the objective function in row 10 enter the equation.

Step 1-3				
Variable values to manipulate	0	0		
Name of Decision Variable	x	y	Sum	RHS
Objective to maximize	360	800	0	
Available time on M1	2	8	0	60
Available time on M2	5	2	0	60

Step 4: Activate the solver. To do this, go to Tools in the toolbar and click on Solver. The following dialogue then appears:



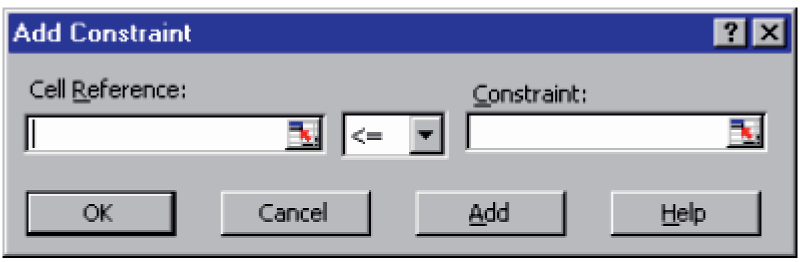
Step 5: Define where the objective function is by defining the variable dialogue box called **Set Target Cell** as the cell number where you added the objective function up.

Step 6: Choose whether to maximize or minimize using the buttons just below the Set Target Cell box.

Step 7: Identify the decision variables by entering the range in which they fall in the **By Changing Cells box.**

Step 8: Enter consideration of the constraint equations into the model by clicking the Add

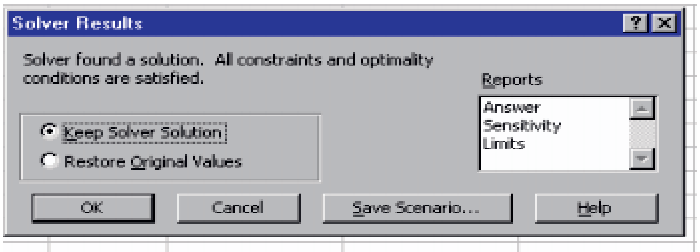
button to the right of the Subject to the Constraints box. You then get the dialogue below;



This step is multistep that depends on the number of constraints.

Step 9: Review the problem and when all looks right, solve by clicking the solve button.

Step 10: Excel will solve LP problem based on the formulas you inputted. When Excel finds an optimal solution, the following appears.



Step 11: Choose desired output reports. Highlight both the Answer Report and Sensitivity Report. Click on Keep Solver Solution and OK then the Reports will be generated. You will see that Excel has entered optimal **Decision Levels** and Total Resource Use in proper cells. And added two new sheets one with the answer and the other for sensitivity. For our example we have get the following two results,

Table1. Result in proper sheet

Variable values to manipulate	10	5		
Name of Decision Variable	x	y	Sum	RHS
Objective to maximize	360	800	7600	
Available time on M1	2	8	60	60
Available time on M2	5	2	60	60

Table 2

Microsoft Excel 15.0 Sensitivity Report

Worksheet: [LPP Example 1.xlsm]Sheet1

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Variable Cells

Cell	Name	Final Value	Reduced Gradient
$B$8	Variable values to manipulate x=	10	0
$C$8	Variable values to manipulate y=	5	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
$D$11	Available time on M1 Sum	60	91.11111111
$D$12	Available time on M2 Sum	60	35.55555556

Microsoft Excel 15.0 Answer Report

Worksheet: [LPP Example 1.xlsm]Sheet1

Report Created: 8/8/2021 2:40:01 PM

Result: Solver found a solution. All Constraints and optimality conditions are satisfied.

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.047 Seconds.

Iterations: 0 Sub-problems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Central

Max Sub-problems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume Non-Negative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$10	Objective to maximize Sum	7600	7600

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$8	Variable values to manipulate x=	10	10	Contin
$C$8	Variable values to manipulate y=	5	5	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$11	Available time on M1 Sum	60	$D$11<=$E$11	Binding	0

$D$12	Available time on M2 Sum	60	$D$12<=$E$12	Binding	0

Exercise 4.13

Solve the Linear Programing Problem by using Excel solver or spreadsheet.

a. Maximize $z = 4x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 0,$
$x_1 \leq 4,$
$x_1 \geq 0, \;\; x_2 \geq 0.$

b. Maximize $z = 5x_1 + 6x_2$

Subject to

$3x_1 + 4x_2 \leq 18,$
$2x_1 + x_2 \leq 7\,,$
$x_1 \geq 0\,, \;\; x_2 \geq 0.$

c. Minimize $z = 2x_1 + 3x_2$

Subject to

$4x_1 + 2x_2 \geq 12,$

$x_1 + 4x_2 \geq 6\,,$

$x_1 \geq 0, \;\; x_2 \geq 0.$

Summary

- Half-Plane: The region on a side of a line in the xy-plane.
- A system of linear inequalities is a collection of two or more linear inequalities to be solved simultaneously.
- A graphical solution of a system of linear inequalities is the graph of all ordered pairs (x, y) that satisfy all the inequalities.
- A point of intersection of two or more boundary lines of a solution region is called a vertex (or a corner point) of the region.
- A number $M = f(c)$ for some c in $I = \{x : a \leq x \leq b\}$ is called the maximum value of f on I, if $M \geq f(x)$, for all x in I.
- A number $m = f(d)$ for some d in $I = \{x : a \leq x \leq b\}$ is called the minimum value of f on I, if $m \leq f(x)$, for all x in I
- A value which is either a maximum or a minimum is called an optimum (or extremum) value.
- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called objective function) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).
- Variables are sometimes called decision variables and are non-negative. A few important linear programming problems are:
  - i) Diet problems
  - ii) Manufacturing problems
  - iii) Transportation problems
  - iv) Allocation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.
- Any point outside the feasible region is an infeasible solution.

- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

Review Exercise

1. Determine whether each ordered pair is a solution to the inequality $y > x - 1$.

   a. $(0, 1)$ b. $(-4, -1)$ c. $(-2, 0)$

   d. $(4,2)$ e. $(3, 0)$

2. Graph each of the following linear inequalities.

   a. $y \leq x - 1$ b. $y < \frac{3}{5}x + 2$

   c. $y \leq -\frac{1}{2}x + 4$ d. $y \geq -\frac{1}{3}x - 2$

3. Solve each system by graphing.

   i. $\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$ ii. $\begin{cases} 2x + 4y > 4 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$

   iii. $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$ iv. $\begin{cases} 2x + y > -6 \\ -x + 2y \geq -4 \end{cases}$

4. Solve the inequality $\begin{cases} y \leq -5x + 2, \\ x \geq 0, \\ y \geq 1. \end{cases}$

5. Discuss steps of solving linear programming problems.

6. Sultan works two part time jobs. One is at a gas station that pays Birr 11 an hour and the other is Computer technician for Birr 16.5 an hour. Between the two jobs, Sultan wants to earn at least Birr 3300.00 in four weeks. How many hours does Sultan need to work each job to earn at least Birr 3300. Assume Sultan has 15 working hours per day and works 5 days per week. Write an inequality which models the situation. Graph the inequality and list three solutions to the inequality.

7. Fikadu was told by his Doctor that he needs to exercise enough to burn 500 calories each day. Thus, he prefers either to run or bike and burns 15 calories per

minute while running and 10 calories a minute while biking. He can allocate at most 40 minutes per day for exercises. Find the inequality that models the situations. Graph the inequality and list three solutions to the inequality.

8. Find the maximum and minimum values of the objective functions subject to the given constraints.

i) Objective functions $z = 6x + 4y$

Subject to $\begin{cases} x + y \le 5 \\ y \ge 2 \\ x \ge 0, y \ge 0 \end{cases}$

ii) Objective functions $z = 3x + 5y$

Subject to $\begin{cases} x - 2y \le 6 \\ x \le 10 \\ y \ge 1 \\ x \ge 0, y \ge 0 \end{cases}$

iii) Objective function

$z = 6x + 4y$

Subject to:

$\begin{cases} 2x + 3y \le 30 \\ 3x + 2y \le 24 \\ x + y \ge 3 \\ x \ge 0, y \ge 0 \end{cases}$

iv) Objective function

$z = 3x + 9y$

Subject to: $\begin{cases} x + 4y \le 8 \\ x + 2y \le 4 \\ x \ge 0, y \ge 0 \end{cases}$

9. Maximize or Minimize, $Z = 3x + 4y$,

Subject to: $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \le 2 \\ x \ge 0, y \ge 0 \end{cases}$

10. Minimize and Maximize $Z = 5x + 10y$,

Subject to: $\begin{cases} x + 2y \le 120 \\ x + y \ge 60 \\ x - 2y \le 0 \\ x \ge 0, y \ge 0 \end{cases}$

11. Minimize and Maximize $Z = x + 2y$,

Subject to: $\begin{cases} x + 2y \le 100 \\ 2x - y \le 0 \\ 2x + y \le 200 \\ x \ge 0, y \ge 0 \end{cases}$

12. Maximize, $Z = -x + 2y$,

Subject to the constraints: $\begin{cases} x \geq 3 \\ x + y \geq 5 \\ x + 2y \leq 6 \\ y \geq 0 \end{cases}$

13. Maximize $Z = x + y$,

Subject to $\begin{cases} x - y \leq -1 \\ -x + y \leq 0 \\ x \geq 0, y \geq 0 \end{cases}$

14. Find the maximum and minimum values of

a. Objective function $Z = 6x + 10y$

Subject to $\begin{cases} 2x + 5y \leq 10 \\ x \geq 0,\ y \geq 0 \end{cases}$

b. Objective function $Z = 4x + y$

Subject to: $\begin{cases} x - y \leq 40 \\ 2x + 3y \leq 72 \\ x \geq 0,\ y \geq 0 \end{cases}$

15. A health fitness center wants to produce a guide to healthy living. The center intends to produce the guide in two formats: a short video and a printed book. The center needs to decide how many of each format to produce for sale. Estimates show that no more than 10,000 copies of both items together will be sold. At least 4,000 copies of the video and at least 2,000 copies of the book could be sold, although sales of the book are not expected to exceed 4000 copies. Let x be the number of videos sold, and y the number of printed books sold.

a. Write down the constraint inequalities that can be deduced from the given information.

b. Represent these inequalities graphically and indicate the feasible region clearly.

c. The health fitness center is seeking to maximize the income, I, earned from the sales of the two products. Each video will be sold for Birr 50 and each book for Birr 30. Write down the objective function for the income.

d. What maximum income will be generated by the two guides?

e. Solve the problem by using spreadsheets solver.

16. A housing corporation plans to build a maximum of 11 new houses in a large city. This company will build these houses in one of three sizes for each location – a convenience house, standard house, and luxury house. The convenience house requires Birr 4.125 million to build and 30 employees to operate. The standard house requires Birr 8.25 million to build and 15 employees to operate. The luxury house requires Birr 12.375 million to build and 45 employees to operate. The corporation can dedicate Birr 82.5 million in construction capital, and 300 employees to staff the stores. On the average, the convenience house nets Birr 1.2 million annually, the standard nets Birr 2 million annually, and the luxury nets Birr 2.6 million annually. How many of each should they build to maximize revenue?(Challenging, extend your excel knowledge to three variables)

17. To meet the requirements of a specialized diet a meal is prepared by mixing two types of cereal, Wheat and Sorghum (Mashila). The mixture must contain x packets of Wheat cereal and y packets of Sorghum cereal. The meal requires at least 15 g of protein and at least 72 g of carbohydrates. Each packet of Wheat cereal contains 4 g of protein and 16 g of carbohydrates. Each packet of Sorghum cereal contains 3 g of protein and 24 g of carbohydrates. There are at most 5 packets of cereal available.
    i) Write down the constraint inequalities.
    ii) Identify the feasible region.
    iii) If Wheat cereal costs Birr 40 per packet and Sorghum cereal also costs Birr 40 per packet, use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a minimum.
    iv) Use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a maximum (give all possibilities).

UNIT 5

MATHEMATICAL APPLICATIONS IN BUSINESS

Financial Center of Ethiopia and the new Head Quarter of CBE

Unit Outcomes

By the end of this unit, you will be able to:

- Know common terms related to business.
- Know basic concepts in business.
- Understand time value of money.
- Apply mathematical principles and theories to practical situations.

Unit Contents

KEY Words

- Amortization
- Annual interest rate
- Appreciation
- Checking accounts
- Compound proportion
- Compound interest
- Constant of variation
- Continued ratio
- Creditor
- Debt financing
- Debtor
- Depreciation
- Equity financing
- Equivalent annual rate
- Equivalent ratios
- Financing
- Future value of an annuity due
- Future value annuity factor
- Inverse proportion
- Investing
- Investment
- Markup interest
- Ordinary annuity
- Percentage
- Present value of ordinary annuity
- Principal
- Proportion
- Rate
- Ratio
- Saving
- Savings accounts
- Simple interest
- Straight-line depreciation
- Term of loan
- Tax

Introduction

Mathematics applications in business are designed to provide a sound introduction to the uses of mathematics in business and personal finance applications. It has dual objectives of teaching both mathematics and financial literacy. The unit wraps each skill or technique it teaches in a real-world context that shows you the reason for the mathematics you're learning.

This unit has four sections. The first section deals with the concept of ratio, rate, proportion, and percentage. Here, you will see how these concepts are implemented in business. The second section deals with the time value of money and computation of simple and compound interests, annuity, amortization and depreciation of a fixed asset. The third section deals with the concepts of saving, investing, and borrowing money. The last section addresses taxation and the different types of taxes commonly implemented in Ethiopia. Each section accommodates solving problems that are associated with business activities and uses spreadsheets for selected computations.

5.1 Basic Mathematical Concepts in Business

The concepts of ratio, rate, proportion and percentage are widely used whenever we deal with business in our daily life activities.

In this section, you will learn how ratios, rates, proportions and percent, are used in many everyday problems.

Activity 5.1

When S Super market offers a 25% discount on vegetables and Y Super market advertises "Buy four, get one free". The price-conscious shopper must decide which one is the better offer if she intends to buy five packages of vegetables.

5.1.1 Ratio

A ratio, which is a comparison of two numbers by division, is the quotient obtained if the first number is divided by the second number. Since a ratio is the quotient of two numbers divided in a definite order care must be taken to write each ratio in intended order.

For example, the ratio of 6 to 1 is written

$$\frac{6}{1} \text{ (as a fraction) } \text{ or } 6:1 \text{ (using a colon).}$$

In general, the ratio of a to b can be expressed as

$$\frac{a}{b} \text{ or } a \div b \text{ or } a:b$$

To find the ratio of two quantities, both quantities must be expressed in terms of the same unit of measures before their quotient is determined.

For example to compare the value US dollars to Japanese Yen, we first convert Yen to dollars and then find the ratios, which is 1:103. Therefore, 1 dollar is worth 103 times as much as Japanese Yen. The ratio has no unit of measurement.

Example 1

What is the ratio of 2 miles to 1 kilometer?

Solution

To compare two measurements in different units you must change one of the units of measurement to the other unit or both to another standard unit.

1 mile is equal to 1.6 km. So, 2 miles =3.2 km

Therefore, the ratio is 3.2:1 or 1mile =1600meter and 1km=1000meter;

The ratio of 2 miles to 1 kilometer becomes $\frac{3200}{1000} = \frac{16}{5}$ *or* $16:5$ *or* $3.2 : 1$.

A ratio can be expressed in one of two ways:

i. Part-to-whole ratio or ii. part-to-part ratio

Example 2

The following table gives the number of teachers in a given school according to their education level and sex.

a. What is the ratio of female diploma holders to the number of teachers in the school?

b. What is the ratio of diploma holders to degree holders in the school?

	Diploma Holders	Degree Holders	Total
Male	26	46	72
Female	16	12	28
Total	42	58	100

Solution

a. The first question is asking the part-to-whole ratio, hence it is 16:100 or 4:25.

b. The second question is asking the part-to-part ratio, hence it is 42:58 or 21:29.

Activity 5.2

In Ethiopia, bank branches to population ratio was 1:83195 in 2010. But this ratio has been changed to 1: 17732 in 2019. Discuss if this ratio has changed.

Exercise 5.1

1. There are a total of 200 pupils in a certain art school. From those 25 students are major in music and the remaining major in sport. What are the ratio of the students major in music and sport?
2. A profit of Birr 19,560 is to be divided between two partners in the ratio of 3: 1. How much should each receive?
3. Two individuals' shares in a certain company are in the ratio is 10: 1. How many times is the number of shares owned by the major shareholder greater than the number owned by the minor shareholder?

Equivalent ratios

When a ratio of two numbers such as $\frac{24}{16}$ is a fraction, we divide the numerator and denominator by the same nonzero number to find equivalent ratio. For example,

$\frac{24 \div 2}{16 \div 2} = \frac{12}{8}$ $\quad \frac{24 \div 4}{16 \div 4} = \frac{6}{4}$ $\quad \frac{24 \div 8}{16 \div 8} = \frac{3}{2}$

If a ratio is expressed in simplest form, then both terms of the ratio are whole numbers. But when there is no whole number other than 1 that is a factor of both of these terms. Hence, $\frac{24}{16}$ in simplest form is $\frac{3}{2}$.

Continued Ratio

Activity 5.3

In concrete the ratio of cement, sand and gravel is 1:3:2. Assume cement is measured in a bag of 50kg, sand and gravel measured in cubic meter. What does this ratio mean? Discuss in a group.

Comparisons can also be made for three or more quantities. A continued ratio is a comparison of three or more quantities in a definite order.

The length of rectangular box is 75 centimeters, the width is 60 centimeters and the height is 45 centimeters. The ratio of length to width is $75:60$ and width to height is $60:45$. Then we can write these ratios as continued ratio $75:60:45$. Here, the measure of length(l), width(b) and height(h) of the rectangular box in simplest form is $5:4:3$.

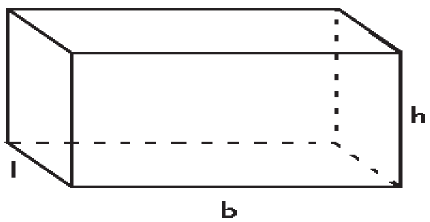


Figure 5.1

Remark: In determining the ratio of certain parameters unit of measurement must be the same.

Example 3

Allocate Birr 1500 in the continued ratio $2:3:7$.

Solution

Note that the terms in the ratio are positive integers. First, you need to determine the total number of parts to be allocated. That is $2 + 3 + 7 = 12$.

Now determine the value of each single part, which is obtained by dividing the total amount by the total parts to be allocated: $\frac{1500}{12} = 125$ Birr.

To allocate, multiply each term of the ratio by the value of the single part, i.e.

$2 \times 125 = 250,\ 3 \times 125 = 375,$ and $7 \times 125 = 875.$

Therefore, money will be shared in the quantities Birr 250, Birr 375, and Birr 875, respectively.

Example 4

The ratio of the angles of a quadrilateral are in the ratio, $\mathbf{7{:}\,9{:}\,10{:}\,10.}$ What is the measure of the angles of a quadrilateral?

Solution

Since the sum of the angles of a quadrilateral is 360^0. By adding $7 + 9 + 10 + 10 = 36$. Dividing 360 by 36 we get 10. It follows that the angles of the quadrilateral are $70^0, 90^0, 100^0, 100^0$.

Example 5

The Grand Ethiopian Renaissance Dam (GERD) which has a capacity of 74 billion metric cube contains 18.4 billion cubic meters of water in two round filling. That is, 4.9 billion cubic meters in the first round filling and 13.5 billion cubic meters in the second round filling as of July 2021.

a. Find the ratio of the water filled in the dam in the first round to the total capacity of the dam.

b. Find the ratio of the water filled in the dam in two rounds to the total capacity of the dam.

c. How full the dam is?

Figure 5.2. GERD in Guba, Ethiopia, on February 19, 2022.

Solution

Let F_1 be the first round filling, F_2 is second round filling, and F is the cumulative in the first n round filling. Setting C is the total capacity

a. $F_1 = 4.9$ Billion cubic meter, $C = 74$ billion cubic meters, $F_1 : C$ implies that 4.9:74. In the first round 1/15 the dam was filled

b. $F = F_1 + F_2 = 18.4$ Billion cubic meters, $F : C$ implies $18.4 : 74$. In the first two rounds about one fourth of the total capacity filled by water.

c. The ratio of 18.4:74 is approximately 1:4, therefore, one quarter of the dam is full

Exercise 5.2

1. A sum of money was divided between Debora, Kalid, and Mesfin in the ratio of $5:3:1$, respectively. Debora has received Birr 3504. How much money was there to start with?
2. A block of wood whose length is twice as its width and its height is one-half its width. Write, in the lowest term, the ratio of length to width to height.
3. Express each of the following ratio in the simplest form:

   a. $1\frac{1}{2}$ hour to 30 minutes.

   b. 6 Birr to 50 cents

   c. 1 kilometer to 1 meter

4. A legendary athlete Haile Gebresilassie ran twelve 10,000m international games and won 7 gold, one silver, 2 bronze and certificates for one 5^{th} and one 6^{th} places between 1992 to 2008. What is the ratio of the number of games he won Gold to the number of games he ran?
5. A chemist wishes to make 15 liters of an acid solution by using water and acid in the ratio 3:2. How many liters of water and acid does she use?

5.1.2 Rates

Activity 5.4

A combine harvester machine can harvest three hectares of wheat field in one hour at a rate of 150 Birr per hour. If a farmer owns 16.5 hectares of wheat field how much does he pay to harvest his wheat?

A rate, like a ratio, is a comparison of two quantities. But if the quantities have different units of measures, their ratio has a unit of measure.

For Example, if an automobile travels 60 kilometers in 1 hour, its rate of speed is a ratio that compares the total distance covered to the time that the automobile took to travel.

$$\text{rate} = \frac{\text{distance traveled}}{\text{time taken}} = \frac{60}{1}\,\text{km/h} = 60\ \text{km/h}$$

A rate that has a denominator of 1 is called a unit rate. A rate that identifies the cost of an item per unit is called the unit price. For Example, 26 Birr/kg of sugar or 50 Birr/liter of Milk are unit prices.

Example 1

Adane scored 12 goals in 4 Ethiopian premier league games. Express, in lowest terms, the average rate of the number of goals Adane scored per games.

Solution

$$\text{rate} = \frac{12 \text{ goals}}{4 \text{ games}} = \frac{3 \text{ goals}}{1 \text{ game}} = 3 \text{ goals per game}$$

Adane scored an average rate of 3 goals per game.

In dealing with business, production, population size, and so on, it is common to describe the amount in which certain quantity has increased or decreased based on some starting point or fixed level. This will the rate of change of a given quantity given by:

$$\text{rate of Change} = \frac{\text{amount of change}}{\text{original amount}} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}}$$

The rate of change be a rate of **increase** if the amount of change is **positive** and a rate of **decrease** if the amount of change is **negative**.

Example 2

The average price of Yirgacheffe coffee for November, 2018/19 was about 1,277 Birr per 17kg (Feresula). In the 2019/20 harvest season (November), Yirgacheffe coffee was traded on average of about 1,780 Birr per 17kg (Feresula). What is the rate of change in the price of one kilogram of coffee from 2018/19 to 2019/20 harvest season?

Solution

We are given that the original price = Birr 1277 per 17 kg and the new price = Birr1780 per 17 kg. Hence change in price = Birr 1780 – Birr 1277 = Birr 503 per 17 kg

$$\text{rate of change} = \frac{\text{amount of change}}{\text{original amount}} = \frac{503}{1277} = 0.394.$$

The rate change of the price of coffee is 0.394 on the given time interval.

Example 3

Suppose the daily income a bakery gets from the sale of a special bread increases from Birr 120,000 to Birr 171,000. At the same time the number of special bread sold increases from 1000 to 2000. What is the average change in income per additional special bread that was sold?

Solution

:Average Rate of Change $= \frac{\text{amount of change}}{\text{change in unit}} = \frac{171000-120000}{2000-1000} = \frac{51000}{1000} = 51$

So the average increase in revenue, per additional bread sold, is Birr 51.

Exercise 5.3

1. Yohannes and Semira use computers for word processing. Yohannes can type 1215 words in 15 minutes while Semira can type 2700 words in 30 minutes. Who is the faster typist?
2. A carpenter's daily production of chairs increased from 20 units to 40 units. At the same time his daily income (or revenue) increased from 1600 Birr to 2400 Birr. As a result of this increase, what is the rate of change of income per unit?
3. A steel company imported 70 tons of raw materials from India in 2010. In 2014 the company imported 105 tons of raw materials from the same country. What is the rate of change of amount imported?

5.1.3 Proportion

A **proportion** is an equation that states that two ratios are equal. Since the ratio $5:30\ or\ \frac{5}{30}$ is equal to the ratio1$:6\ or\ \frac{1}{6}$, we may write the proportion

$$5:30 = 1:6 \text{ or} \frac{5}{30} = \frac{1}{6}.$$

Each of these proportion reads as " 5 is to 30 as 1 is to 6." For $a, b, c, d \in \mathbb{R}$, with $b \neq 0$ and $d \neq 0$, one way of denoting a proportion is $a : b = c : d$, which is read as "*a* is to *b*" is equal to "*c* is to *d*."

Of course, by definition, $\frac{a}{b} = \frac{c}{d}$ means that a proportion is an equation between two ratios. In the proportion, $a : b = c : d$, with $b \neq 0$ and $d \neq 0$, the four numbers are referred as the **terms** of the proportions. The first and the last terms *a* and *d* are called **the extremes**; the second and third terms *b* and *c* are called the **means**. In the proportion, $a{:}b = c{:}d$, the product of the extremes is equal to the product of the means; that is,

$\frac{a}{b} = \frac{c}{d}$ is equivalently represented as $a \times d = b \times c$.

For three quantities a, b and c such that $\frac{a}{b} = \frac{b}{c}$ which is equivalent to $b^2 = ac$ is called the **mean proportional** between *a* and *c*.

Example 1

On a residence house construction plan of Ato Admasu, 1 cm on the paper represent 25 meters on the ground. Find the distance on the ground such that the distance represented by 3.20 cm on the paper.

Solution

On the map we have the ratio 1cm to 25m. Converting meter to centimeters, 25m=2500cm. Let *x* be the distance on the ground. Then the distance represented by 3.20 cm on the plan can be found by the proportion

$$\frac{1\text{cm}}{2500\text{cm}} = \frac{3.2\text{cm}}{x},$$

$$x = \frac{3.2\text{cm} \times 2500\text{cm}}{1\text{cm}} = 8000 \text{ cm}.$$

Hence, 3.2 cm on the paper is equivalent to 80m on the ground.

Compound proportion

From the above discussion you have seen how change in the quantity of one variable depends on a change in the quantity of another variable (i.e., simple proportion). However, the value of a quantity of the variable most often depends on the value of two or more other variable quantities. For example,

- The cost of sheet metal depends on the area of the sheet, thickness of the sheet, and the cost per unit area of the metal.
- The amount of interest obtained depends on the amount of money deposited in a bank, length of time it is deposited, and rate of interest per year.
- The amount of product produced depends on the amount of capital and labor hour units used.

Definition 5.1

A compound proportion is a situation in which the quantity of one variable depends on two or more other quantities of other variables. Specifically, if a variable quantity y is proportional to the product of two or more variable quantities, we say that y is jointly proportional to these variable quantities, or y varies jointly as these variables.

Two quantities x and y are said to be in direct proportion if they increase or decrease together in such a manner the ratio of their corresponding values remains constant. That is, $\frac{x}{y} = k$, for some positive constant k. Hence, k is a constant of variation.

Two quantities x and y are said to be an inverse proportion if an increase in x causes a proportional decrease in y and vice-versa, in such a manner that the product of their corresponding values remains constant. That is, $xy = k$ where k is a positive number, if x and y are in inverse proportion.

When two quantities x and y are in direct proportion (or vary directly), they are written as $x \propto y$. Symbol " $\propto$ " stands for 'is proportional to'. When two quantities x and y are in inverse proportion (or vary inversely) they are written as $x \propto \frac{1}{y}$.

Example 2

Under the condition that the temperature remains constant, the volume of gas is inversely proportional to its pressure. If the volume of gas is $4m^3$ at a pressure of 1 kilopascal, then what will be the pressure of the gas, if its volume is $3m^3$at the same temperature?

Solution

By Boyle's law, pressure and volume of a gas are inversely proportional at constant temperature. Let the required pressure be x.

Volume of gas in meter cube	4	3
Pressure of gas in kilopascal	1	x

The relationship for Boyle's Law can be expressed as follows: $P_1V_1 = P_2V_2$, where P_1 and V_1 are the initial pressure and volume values, and P_2 and V_2 are the values of the pressure and volume of the gas after change.

So $4 \times 1 = 3 \times x$, Therefore, $x = 4/3 = 1.33$. The volume decreased to 3cubic meter then the pressure increased to 1.33 kilopascal.

Example 3

The table gives pairs of values for the variables x and y.

x	1	2	3	7	?
y	9	18	27	?	900

a. Show that one variable varies directly with the other variable.

b. Find the constant of variations by comparing y to x.

c. Express the relationship between the variables in a formula.

d. Find the values missing in the table.

Solution

$\frac{x}{y} = \frac{1}{9}$ $\frac{x}{y} = \frac{2}{18} = \frac{2 \div 2}{18 \div 2} = \frac{1}{9}$ $\frac{x}{y} = \frac{3}{27} = \frac{3 \div 3}{27 \div 3} = \frac{1}{9}$.

a. Since all the given pairs of values have the same ratio, x and y vary directly.

b. Constant of variation $= \frac{9}{1} = 9$.

c. Write by including the constant of variation 9. That is, $\frac{y}{x} = 9$, $y = 9x$.

d. Solve the equation in c. when $x = 7, y = 9 \times 7 = 63$, when $y = 900$,

$$900 = 9x,$$

$$x = \frac{900}{9} = 100.$$

Exercise 5.4

1. Both x and y vary directly with each other. Let x is 10, y is 15, then, write four corresponding pairs of values of x and y.
2. Both x and y vary inversely with each other. When x is 10, y is 6. Write four corresponding pairs of values of x and y.
3. If the price of 15 postal stamps is Birr 60, then what is the price of 72 postal stamps?
4. If a deposit of Birr 2,000 earns an interest of Birr 500 in 3 years, how much interest would a deposit of Birr 36,000 earn in 3 years with the same rate of simple interest?
5. The mass of an Aluminum rod varies directly with its length. If a 16 cm long rod has a mass of 192 g, find the length of the rod whose mass is 105 g.

5.1.4 Percentage

Activity 5.5

Complete the following table without **using** calculators.

percentage	250	100	625	2000
10%				
25%				
53%				
15%				
7.2%				

Problems dealing with discounts, commissions and taxes involve percentages. A percentage is the numerator of a fraction whose denominator is 100, which is a ratio of a number to 100. The term percent is denoted by % which means "per one hundred". For example, 7% (read as 7 percent) is a ratio of 7 to 100. A percent can be expressed as a fraction or decimal:

$$7\% = \frac{7}{100} = 0.07$$

If an item is taxed at a rate of 15%, then a 100 Birr book costs an additional 15 Birr for tax. Here, three quantities are involved:

- The base, the sum of money being taxed, is Birr 100.
- The rate, or the rate of tax is 15% or 0.15.
- Percentage, or the amount of money being taxed, is 15 Birr.

These three related terms may be written as a proportion: $\frac{percentage}{base} = rate$ or as a formula: $Percentage = base \times rate$

Example 1

Find the amount of turnover tax on a 400,000 Birr sale when the turnover tax rate is 2%.

Solution

Let percentage or amount of tax= $base \times rate = 400{,}000 \times 0.02 = 8000$. The amount of tax is 8,000 Birr.

Example 2

Ethio Telecom offers a discount of 25% off by using Tele Birr purchase of its products. What is a regular price of a product that a customer purchased for 81 Birr?

Solution

The rate of the discount is 25%. Therefore the customer paid (100-25)% or 75% of the regular price. The percentage is given as Birr 81, the base is unknown. Let n=regular price or base.

Method I: Use proportion:

$$\frac{\text{percentage}}{\text{base}} = \text{rate}$$

$$\frac{81}{n} = \frac{75}{100}$$

$$81 = 0.75n, \quad n = \frac{81}{0.75} = 108$$

Method II: Formula Percentage=base×rate= $n \times 0.75 = 81$,

$n = \frac{81}{0.75} = 108$. Hence, the regular price of the product was 108 Birr.

Check: If 25% of 108 is subtracted from 108 does the difference equal 81?

$$0.25(108) = 27$$

$108 - 27 = 81.$ (Correct)

Example 3

What is the meaning of the percentage symbol in road? What angle the percentages correspond to in road sign? See Figure 5.3 and Figure 5.4.

Figure 5.3 Percentage sign in road

Trigonometric in real life.

What angle the percentages correspond to in road sign?

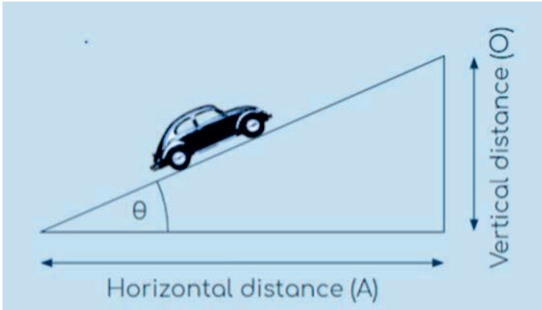


Figure 5.4

Solution

The sign 10% means that the vertical change is 10% of the horizontal change: that is, for every 10 meter travel horizontally there is 1 meter up. In general a bigger percentage translates as a steeper hill.

Calculating the slope of a straight line, as would be taught in a lower class ("rise over run" or "change in y divided by change in x") is equivalent to calculating, $\tan\theta = \frac{O}{A}$. Therefore, 10% means $\tan\theta = 0.1$. θ is the tangent inverse of 0.1. In our example 10% is the same as an angle 5.71^0C as shown in Figure 5.3.

Exercise 5.5

1. In (i) - (iv), find each indicated percentages.
   - i. 2.5% of 400
   - ii. 12.5% of 175
   - iii. 35% of 1500
   - iv. 100% of 77
2. Find each number or base.
   - a. 8% of what number is 12?
   - b. 75% of what number is 24?
   - c. 33% of what number is 200?
   - d. 15% of what number is 1.91?
3. How much silver is found in 80kg of an alloy that is 8% silver?

4. Last week, Bontu answered 30 out of 36 questions correctly on a test. This week she answered 20 out of 24 questions correctly. On which test did she have better result?
5. The monthly salary of Mr. Tilahun is 26% more than Mr. Jemal salary. Assume Mr. Jemal salary is Birr 15396.00. What is the salary of Mr. Tilahun?
6. What does 25% mean in road symbol? What is the slope of the road in degree?

Markup

In order to make a profit, any institution or company must sell its products for more than the product costs the company expended to produce or buy the product. The difference between a product selling price and its cost is called **markup**.

Markup = selling price − cost price.

Example 4

The price of cement is Birr 250 per quintal. If you sell it for Birr 330 per quintal, what is the markup per quintal?

Solution

Markup = Selling price − Cost, thus,

$$\text{Birr } 330 \text{ per quintal} - \text{Birr } 250 \text{ per quintal} = \text{Birr } 80 \text{ per quintal.}$$

Markup is usually expressed in terms of percentage with respect to selling price and cost.

Markup with respect to selling price is given by;

$$\text{Markup percent} = \frac{\text{markup}}{\text{selling price}} \times 100\%.$$

Similarly markup with respect to cost is given by:

$$\text{Markup percent} = \frac{\text{markup}}{\text{cost}} \times 100\%.$$

Example 5

A boutique bought a T-shirt for Birr 546 and wanted a markup of 30% on retail. What is the selling price?

Solution

Given cost = Birr 546. Markup percent is 30% on selling price. Then we need to find selling price.

Cost = selling price – markup = $(100 - 30)\% = 70\%$ of selling price.

This is called the complement of markup percent on selling price.

Hence, the selling price will be cost = $0.70 \times selling\ price$

$$\Rightarrow 546\ Birr = 0.70 \times selling\ price$$

$$\Rightarrow selling\ price = \frac{546}{0.7} = 780\ Birr.$$

Discount

Activity 5.6

Sometimes retailers and manufacturing company offer a discount. Why do those companies offer a discount?

Discount is a reduction on the original selling price. **Regular price** is the price at which an article is offered for sale. **Sale price** is the price that the customer pays in the case of a discounted article. The sale price is obtained by deducting the discount from regular price.

Example 6

In Ethiopian New Year holiday Ethio telecom offer 20% discount on international voice call. The regular (original) price is Birr 6.00 per minute. Calculate the call price.

Solution

call price = regular price − discount = $6-0.2\times6=6-1.2=Birr\ 4.80$ per minute.

$$Discount\ rate=\frac{disount}{regular\ price}\times100\%$$

Example 7

On festive shopping the price of edible oil is 20% discount of Birr 175.00 per liter. Calculate the original price.

Solution

To calculate original price by using discount rate formula,

$$Discount\ rate=\frac{discount}{regular\ price}\times100\%$$

$$0.2=\frac{discount}{regular\ price}=\frac{175}{regular\ price}$$

$$regular\ price=\frac{175}{0.2}=875$$

Hence, the original price of the edible oil is Birr 875.00 and its selling price is Birr 700.00.

Exercise 5.6

1. A car dealer reduces the price of last year's models at the clearance sale of the end of the year, by a certain amount. If a certain four-door model has been sold at a discounted price of Birr 510,000, with a discount of Birr 90,000, what is the percentage of the discount?
2. One year, a furniture manufacturing firm made a profit of 520,000 Birr. This represented 8% of the volume of the business for the year. What was the volume of the business in that year?

3. A proposal made in the regional council members to raise the minimum wage from Birr 12 to Birr 15 an hour. What was the proposed percent of increase to the nearest tenth of a percent?
4. A pair of shoes that costs Birr 110 is sold for Birr 155. Find the markup and the percent of the markup based on the retail (selling price).
5. What is the percent markup on cost if the markup on retail is 37%?
6. Abebe sold a quintal of Teff at Birr 5250 with 45% markup on selling price. Find the cost.
7. Find the percent markup on cost, if markup of the selling price is 30%.
8. Calculate the discount and selling price for a Birr 1700.00 wheat offered at 18% discount.
9. Calculate the regular price of a television set, assuming that a 25% discount would have been worth Birr 2,000.00.

5.2 Time Value of Money

For living in someone's house or using properties, individuals return the cost of the service they were given in terms of money. The money that they pay in return is known as **rent.**

Now, suppose someone needs to borrow Birr 200, and you agree to lend him. If he offered to pay you back the full Birr 200 for one year, would you agree such a term? It hardly seems fair because you wouldn't get any other compensation. Just as in renting the house, even though you will eventually get your property back, it only seems fair if you get some benefit for allowing the use of what belongs to you.

We ordinarily call the payment for the temporary use of property such as houses, apartments, equipment, or vehicles a rent though we don't normally use that term in the case of money. Instead we call that payment for use of money an interest. Interest is either simple or compound.

5.2.1 Simple Interest

Definition 5.2

Interest is what a borrower pays a lender for the temporary use of the lender's money. In other words, it is the "rent" that a borrower pays a lender to use the lender's money.

Interest is paid in addition to the repayment of the amount borrowed. In some cases, the amount of interest is spelled out explicitly. If we need to determine the total amount to be repaid, we can simply add the interest on to the amount borrowed.

From the investment point of view, interest is income from invested capital for investor and expense for the one who used the money or for the borrower. The capital originally invested is called the principal. The principal of a loan is the amount borrowed. The sum of the principal and interest due (or paid) is called the amount (or future value or accumulated value).

There are a few special terms that are used with loans as well.

A **debtor** is someone who owes someone else's money. A **creditor** is someone to whom money is owed. The amount of time for which a loan is made is called its **term**.

For simple interest, the interest is computed on the principal during the whole time, or term of the loan, at the stated annual rate of interest.

To calculate simple interest, we multiply the amount borrowed times the interest rate (as a decimal) times the amount of time.

$$I = Prt \qquad 5.1$$

where, I simple interest, P the principal, r annual interest rate and t term of loan.

Example 1

Ashamo borrows Birr 18,500 at 9.5 % simple interest rate for 2 years. How much interest will he pay?

Solution

The principal $P = 18{,}500$, the interest rate $r = 9.5\% = \frac{9.5}{100} = 0.095$ and the time t= 2 years. So, we begin with our formula $I = Prt = 18500 \times 0.095 \times 2 = 3515$. The amount of interest paid in two years of time is Birr 3515. The total amount of money after two years will be

$$A = P + I = 18500 + 3515 = 22015$$

This amount is known as future value of a simple interest.

The future value of a simple interest

Beginning: $A = P$

At the end of first year: $A = P + I = P + Pr = P(1 + r)$

At the end of second year: $A = P(1 + r) + Pr = P(1 + r + r) = P(1 + 2r)$

Similarly at the end of third year; $A = P(1 + 3r)$.

Finally, after t years the future value of a simple interest A is

$$\boldsymbol{A = P(1 + rt)},$$

where P is the principal, r is the simple interest rate per year, and t is the time in years.

Sometimes interest is paid in situations we might not normally think of as loans. If you deposit money in a bank account, you probably expect to be paid interest even though you probably don't think of your deposit as a loan. In reality, though, it actually is a loan. When you are depositing money to a bank account you are actually loaning that money to the bank.

Example 2

If Birr 2,500 is deposited in Commercial Bank with a simple annual interest rate of 7 %, find the amount of the interest and future value at the end of the fourth month.

Solution

In this example $P = 2500, r = 0.07, t = \frac{4}{12} = 0.33\ year$. Substituting these values in equation (5.1) we obtain,

$$I = Prt = 2500 \times 0.07 \times 0.33 = 57.75$$

The amount of interest will be Birr 57.75. The future value A is

$$A = P + I = 2500 + 57.75 = 2557.75.$$

There are different types of bank deposits. Checking and savings accounts are familiar examples of ways in which you loan money to banks. These are sometimes referred to demand accounts because you can withdraw your money at any time you want (i.e., "on demand"). Another common type of account is a certificate of deposit, or CD or commonly called "fixed time deposit." When you deposit money into a CD, you agree to keep it on deposit at the bank for a fixed period of time.

Example 3

A Saving and Credit Association deposited Birr 13,000,000.00 (Thirteen million) into a 1-year fixed time deposit paying with 13.5% simple interest per annum. How much will his account be worth at the end of the term?

Solution

$$I = Prt = 13000000 \times 0.135 \times 1 = 1{,}755{,}000$$

The Saving and Credit Association gets Birr 1,755,000 interest yearly. At the end of the term, the Saving and Credit Association account will contain both the principal and interest, so the total value of the account will be

$13{,}000{,}000 + 1{,}755{,}000 = 14{,}755{,}000$.

Exercise 5.7

1. Derartu loaned Tesfaye Birr 12,000 for 6 months. Tesfaye paid back Birr 12,600. How much interest did he didrpay? What is the annual interest rate?
2. Gemechu loaned Hagos Birr 800, and 2 years later Hagos will pay back Birr 900. How much total interest will Gemechu receive? What is the annual interest rate?
3. Samir borrowed Birr 7,829.14 for 1 year at a simple interest rate 9.75% annually. How much will he need to repay the loan?

Determining Principal, Interest rate and Time

So far, we have developed the ability to calculate the amount of interest due when we know the principal, rate and time. However, situations may arise where we already know the amount of interest, and instead need to calculate one of the other quantities. For example, consider these situations:

- A retiree hopes to be able to generate Birr 10,000 income per month from an investment account that earns 12 % simple interest. How much money would he need in the account to achieve this goal?
- Tekletsadik borrowed Birr 1500 from his brother-in-law, and agreed to pay back Birr 1650 one year later. What rate of simple interest is Tekletsadik paying for this loan?
- Zehara deposited Birr 9,750 in a savings account that pays 5.25% simple interest per year. How long will it take for her account to grow to Birr 10,000.00?

The first bulleted question is **finding principals**. For the problem monthly income would be $Income = 10{,}000$, rate of interest $r = 12\%$, the time is one month $=\frac{1}{12}$ of a year.

From the Formula, $I = Prt$, dividing both sides by rt, we obtain

$P = \frac{I}{rt} = \frac{10000}{0.12\times(\frac{1}{12})} = 1{,}000{,}000.00$ to earn Birr 10,000 monthly. The retiree must invest Birr 1 million on the stated interest rate.

On the second bulleted problem, principal, future value of money and the term of the loan were given. The required is the **rate of simple interest**.

Similar to the above problem rate of interest is calculated by the formula $r = \frac{I}{Pt}$,

I=1650-1500=150, which is the difference between pay back amount and borrowed amount.

$$r = \frac{150}{1500 \times 1} = 0.1.$$

So the rate is 0.1 or 10%.

On the third bulleted problem is finding the term of the loan or time needed to settle the loan.

$$P = 9750, I = 10000 - 9750 = 250, r = 5\frac{1}{4}\% = 0.0525$$

By dividing the formula $I = Prt$, both sides by Pr, we obtain

$$t = \frac{I}{Pr}$$

Therefore, $t = \frac{250}{9750\times0.0525} = 0.4884$, the number tell us 0.4884 years, which is less than half a year. Multiply by 365 days and we obtain

$0.4884 \times 365 = 178.26$ days. It takes around 179 days to grow to Birr 10,000.

Example 4

Dakito borrowed Birr 2,000 and paid back a total of Birr 2,125 in a year. How much interest did he pay? What is the annual interest rate?

Solution

Dakito paid an interest I, which is the difference between principal and returned amount, i.e. $I = 2{,}125 - 2{,}000 = 125$.

From Equation, $r = \frac{I}{Pt}$, in this example $P = 2{,}000, I = 125$ and $t = 1$. Therefore,

$r = \frac{125}{2000} = 0.0625$. Thus, the annual interest rate is 6.25%.

Example 5

Find the term of a loan if the principal is Birr 20,000, the interest rate is 8%, and the total interest is Birr 320.

Solution

Given $P = 20{,}000$, $r = 8\%$, and interest earned is $I = 320$. Using the formula $I = Prt$, we get $t = \frac{I}{Pr} = \frac{320}{20000 \times 0.08} = 0.2$. The term of the loan can be 0.2 year which is equivalent to $0.2 \times 360 = 72$ days or 2.4 months.

Exercise 5.8

1. Find the total amount that will be required to pay off a 3-year loan of Birr 14,043.43 at 6.09% simple interest per year.
2. Find the principal for a loan if the term is 2 years, the simple interest rate is 9% annually and the interest totals Birr 63.00.
3. How much does Kedija need in her investment account if she wants to be able to receive Birr 450 per month in income from it, assuming a simple interest rate of 7.5% annually?

5.2.2 Compound Interest

"The most powerful force in the universe is compound interest." **Albert Einstein**

Simple interest is very rarely used in real life: almost all banks and other financial institutions use compound interest. This is when interest is added (or compounded) to the principal sum so that interest is paid on the whole amount. Under this method, if the interest for the first year is left in the account, the interest for the second year is calculated on the whole amount so far accumulated.

To begin with, let's consider a Birr 6,000 loan for 5 years at 8% simple interest.

From your previous work you can easily calculate the interest:

$I = Prt = 6000 \times 0.08 \times 5 = 2400.$

And so the maturity value would be $6000 + 2400 = 8400\ Birr.$

This calculation takes us from principal to maturity value without any thought about how the amount of interest grows over the loan's term.

Suppose, instead of just jumping from the Birr 6,000 principal to the Birr 8,400 maturity value 5 years later, let us take a look at the loan year by year along the way.

In the first year, the interest would be

$$I = Prt = 6000 \times 0.08 \times 1 = 480.$$

The principal is still Birr 6,000, the interest rate is still 8%, and the second year is 1 year long, just like the first one. Likewise, the interest in the second, third, fourth, and fifth years would also be the same. Thus under simple interest the loan is growing at a constant rate of Birr 480 per year.

For the case of compound interest, at the end of the first year

$I = \Pr t = 6000 \times 0.08 \times 1 = 480.00$.

The total amount, $A = P + I = 6000 + 480 = 6480$.

At the end of second year, $I_2 = 6480 \times 0.08 \times 1 = 518.40$. As shown in Table 1 below the interest is different yearly. For interest at the end of each years summarized below in Table 1.

Table 1 Simple and Compound interest of Birr 6000 at rate 8% annually

Simple interest of Birr 6000 at rate 8% annually			Compound interest on Birr 6000 at rate 8% annually		
Year	**Interest**	**Balance**	**Year**	**Interest**	**Balance**
start	Not applicable	6000	start	Not applicable	6000
1	480	6480	1	480	6480
2	480	6960	2	518.4	6998.4
3	480	7440	3	559.87	7558.27
4	480	7920	4	604.66	8162.93
5	480	8400	5	653.03	8815.97

Activity 5.7

a. Show how to calculate simple interest and balance in spreadsheet.

b. Show how to calculate compound interest and balance in spreadsheet.

As the balance grows year by year, so does the interest. It is worthwhile to compare the growth of the account year by year under compound interest versus its growth under simple interest.

Table 2 Comparing compound and simple interest of Birr 6000 deposited at a rate of 8% annually.

	Simple Interest		Compound Interest			
Year	Interest	End Balance	Year	Interest	End Balance	Difference in end balance
start	Not applicable	6000	start	Not applicable	6000	0
1	480	6480	1	480.00	6480.00	0.00
2	480	6960	2	518.40	6998.40	38.40
3	480	7440	3	559.87	7558.27	118.27
4	480	7920	4	604.66	8162.93	242.93
5	480	8400	5	653.03	8815.97	415.97

Note that, as time goes by and the account balance grows, so does the impact of compounding. Over the first couple of years, the difference between simple and compound interest is not significant. But as the principal used for the compound interest grows larger and larger, the interest earned on that principal also grows more and more.

Example 1

You deposit Birr 3000.00 in a high-earning account paying 9% compound interest yearly and leave it for three years. What will be the balance on the account at the end of that time?

Solution

Balance after 0 years Birr 3000.00

Interest: 9% of 3000.00 Birr=$0.09 \times 3000 = 270.00$

Balance after 1 year: Birr 3,270.00

Interest: 9% of 3270.00=$0.09 \times 3270 = 294.30$,

Balance after 2 years: Birr 3,564.30.

Interest: 9% of 3564.30=$0.09 \times 3564.3 = 320.78$.

Balance after 3 years: Birr 3,885.08. At the end of third year the depositor will get a compound interest of Birr 885.08.

5.2.2.1 Compound Interest Formula

It should already be clear that for long periods, the year-on-year method of calculating compound interest is somewhat cumbersome, but fortunately there is a formula.

If an initial deposit of P is made at time zero with annual rate $r \times 100\%$, then after the first period the value of the account is $A_1 = P + rP = P(1+r)$, after the second period the value is

$A_2 = A_1 + rA_1 = A_1(1+r) = P(1+r)^2$, and so forth. In general, the value of the account at time n is

$$A_n = P(1+r)^n \qquad 5.2$$

Example 2

Use the compound interest formula to find how much Birr 5,000 will grow to in 5 years at 8% annual compound interest. What will happen in 50 years?

Solution

After 5 years Birr 5000 became an amount. Substituting

$$A = P(1 + \frac{8}{100})^5$$

$$A = 5{,}000(1 + 0.08)^5$$
$$A = 5{,}000(1.08)^5 = 5000 \times 1.469 = 7{,}345.$$

The total compound interest of the period computed by subtracting P from A. That is,

$$A - P = P(1 + \frac{r}{100})^n - P = P[\left(1 + \frac{r}{100}\right)^n - 1].$$

Therefore, compound interest (C.I) is

$$C.I = P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]. \qquad 5.3$$

The interest to be paid in 5 years on Birr 5,000 is $A - P$=7345-5000=2,345. Birr.

In 50 years simply by substitution you get

$$A_{50} = 5000(1 + \frac{8}{100})^{50} = 234508.063.$$

Example 3

Calculate the number of complete years in which a sum of money deposited at 20% compound interest will be doubled.

Solution

Let P be the original amount deposited after n years. This amount grows to doubled, i.e., $2P$.

$A = P(1 + \frac{20}{100})^n = 2P$. Dividing both sides by P you obtain

$(1 + 0.2)^n = 2$. To solve for n you can use your knowledge of logarithmic function.

$$n \log 1.2 = \log 2$$
$$n = \frac{\log 2}{\log 1.2} = 3.8$$

The deposit must be left for 3.8 years but as interest is paid yearly, it would have to be left for 4 years.

Exercise 5.9

1. You deposited Birr 10,000.00 in a high-earning account paying 9.5% compound interest and leave it for five years. What will be the balance on the account at the end of that time? What is the compound interest of the period?
2. For how long must a sum be deposited in an account paying 14% compound interest in order to triple in value?
3. Calculate the annual rate of compound interest that will allow a principal sum to double in value after five years.
4. Calculate the principal which, if deposited at 9.5% compound interest, will grow to Birr 4000 after three years.
5. Ayantu deposited Birr 2,000 at 8% interest compound annually. How many years will it take her to get Birr 3,000?

5.2.2.2 Compounding Frequencies

Activity 5.8

In a group collect information from Bank professionals and savers about:

- How many times in a year does Commercial Bank of Ethiopia calculate interest on deposit of money?
- Choose any bank or credit union and ask how many times they pay interest on annual deposit.

Interest is usually compounded more than once a year. The quoted rate of interest per year is called annual or nominal rate and the interval of time between successive interest calculations is called conversion period or compound period.

As you looked, in previous example, there was no difference between simple and compound interest at the end of the first year. Since no interest was credited to those accounts until the end of the first year, there was no opportunity for interest on interest until the end of the first year. In the second year, interest on interest began to make a difference, and the difference became more pronounced the following year. In this case, interest on interest combined with interest on interest make a difference significantly. What will happen if the interest is calculated monthly or even daily?

What's more, over the course of each year compounding would take place in a total of 12 times, rather than just once. But how much more interest would this monthly compounding mean? If you invested Birr 6,000 at 8% compounded annually, when the first interest is credited at the end of the first year you would have Birr 6,480. Now suppose that we look at the same Birr 6,000 at the same 8%, but this time we compound the interest every month.

The interest earned for the first month would be:

$I = Prt = 6000 \times 0.08 \times \left(\frac{1}{12}\right)$ =Birr 40.

So at the end of the first month the account balance would be Birr 6040.00.

Following the same approach, we find that the next month's interest works out to Birr 40.27—a bit more and continuing on through the rest of the year, we get the result given in Table 3.

Table 3: Compound interest on Birr 6000 compounded monthly at a rate 8%.

Month	Beginning Balance	Interest	Ending Balance
1	6,000.00	40.00	6,040.00
2	6,040.00	40.27	6,080.27
3	6,080.27	40.54	6,120.80
4	6,120.80	40.81	6,161.61
5	6,161.61	41.08	6,202.68
6	6,202.68	41.35	6,244.04
7	6,244.04	41.63	6,285.66
8	6,285.66	41.90	6,327.57
9	6,327.57	42.18	6,369.75
10	6,369.75	42.47	6,412.22
11	6,412.22	42.75	6,454.96
12	6,454.96	43.03	6,498.00

Let's Calculate

$6000 \times 0.08 = 40.00$

Since r is the annual interest rate and the interest is compounded m times per year, the year is divided into m equal conversion periods and the interest rate during each conversion period is $i = r/m$, that is we get interest r/m every $1/m$ years.

Notice that this is Birr 18.00 more than we had at the end of the first year with annual compounding.

Now, if the interest i3s compounded for t years, then there will be $n = mt$ conversion periods in t years. Thus, if you put $n = mt$ and replace r by the expression of interest rate per each conversion period $\boldsymbol{i} = \frac{\boldsymbol{r}}{\boldsymbol{m}}$, we have the future value of compound interest given by:

$$A = P(1 + \frac{im}{100})^{mt} \qquad 5.4$$

Where A is the amount or future value, P is the principal or present value, $\boldsymbol{i} = \frac{\boldsymbol{r}}{\boldsymbol{m}}$, r is annual or nominal rate, t is time in years, and m is the number of conversion periods per year.

In working with problems involving interest, we use the term of payment periods as follows:

- Annually means once a year, i.e. $m = 1$.
- Semi-annually means twice a year, i.e. $m = 2$.
- Quarterly means four times a year, i.e. $m = 4$.
- Monthly means 12 times a year, i.e. $m = 12$ and.
- Daily means 365 times a year, i.e. $m = 365$.

Collectively the value of m implies compounding frequencies, i.e. how many times in a year the interest is calculated on the given deposited amount of money or investment.

Example 1

If Birr 100 is deposited in the Commercial Bank with 10% interest rate annually, find the amount if it is compounded annually, semi-annually, quarterly, monthly, weekly and daily at the end of one year. Assume withdrawal or deposit not allowed on the given interval of time.

Solution

The given values are $P = 100$, at a rate of $r = 10\% = 0.1$ for a period of one year, and compound period of $n = mt$.

a. Annually means $m = 1$, so that the amount at the end of the year is

$$A = 100(1 + \frac{0.1}{1})^1 = 100(1 + 0.1)^1 = 100(1.1) = \text{Birr } 110.$$

b. Semi-annually means $m = 2$, so that the amount at the end of the year is

$$A = 100(1 + \frac{0.1}{2})^2 = 100(1 + 0.05)^2 = 100(1.05)^2 = \text{Birr } 110.25.$$

c. Quarterly means $m = 4$, so that the amount at the end of the year is

$$A = 100(1 + \frac{0.1}{4})^4 = 100(1 + 0.025)^4 = 100(1.025)^4 = \text{Birr } 110.38.$$

d. Monthly means $m = 12$, so that the amount at the end of the year is

$$A = 100(1 + \frac{0.1}{12})^{12} = 100(1 + 0.0083)^{12} = 100(1.0083)^{12}$$

$$= \text{Birr } 110.47.$$

e. Weekly means $m = 52$, so that the amount at the end of the year is

$$A = 100(1 + \frac{0.1}{52})^{52} = 100(1 + 0.0019)^{52} = 100(1.0019)^{52}$$

$$= \text{Birr } 110.51.$$

f. Daily means $m = 365$, for simplified exact method, so that the amount at the end of the year is

$A = 100(1 + \frac{0.1}{365})^{365} = 100(1 + 0.00027)^{365} =$

$100(1.00027)^{365} =$Birr 110.52.

Bankers use $m = 360$, this sometimes known as Bankers rule

$A = 100(1 + \frac{0.1}{360})^{360} = 100(1 + 0.000277)^{360} =$

$100(1.000277)^{360} =$Birr 110.52.

These two methods have no significant difference in two decimal places. You can observe that when the time, principal and interest rates are kept fixed and the number of times the interest is compounded increases, the amount will also increase.

Example 2

Ato Deribe made the following transactions on his account at Bank. Deposited Birr 25,000 on 1st January 2018, withdraw Birr 6,000 on 1st July 2020, deposited Birr 18,000 on 1stJanuary 2019. If the account earns 7.5% interest rate per year compounded monthly, find the balance on the account on 1st January 2022.

Solution

The solution of this example can be calculated by using spreadsheet as follow. As expressed from January 1, 2018 to July 1, 2020 the deposit amount earned interest for 30 month continuously.

Date	Amount	Deposit	Withdrawal	Interest earned	Ending Balance
1-Jan-18	25,000.00			1,940.81	26,940.81
1-Jan-19	26,940.81	18,000.00			44,940.81
31-Dec-19	44,940.81			3,488.87	48,429.69
1-Jul-20	48,429.69		6,000.00	1,844.73	44,274.41
1-Jan-22	44,274.41			5,254.51	49,528.93

Ato Deribe will have Birr 49,528.93 on his account on January 1, 2022.

Exercise 5.10

1. For each of the following combinations of interest rates, compounding frequencies and terms, find the value of i and n that would be used in the compound interest formula.
   a. 8%, quarterly compounding, 10 years
   b. 9%, compounded monthly, 7 years
   c. 15%, compounded semiannually, 15 years
   d. 9%, daily compounding using bankers rule, 8 years

2. Calculate the total amount accumulated if Birr 4,500 is deposited for 7 years at 6% interest compounded monthly.
3. For how long must you leave an initial deposit of Birr 100 in a 12% savings account compounded semi-annually to see it grows to Birr 1200?
4. How much do I need to deposit today into a Certificate of Deposit (CD) paying 13.5% compounded monthly in order to have Birr 100,000 in the account in 3 years?
5. Calculate the principal of an investment that will grow to Birr 600,000.00 after two years compounded quarterly at the annual rate of interest 9%.
6. If Mr. Degalla deposits a sum of money in a bank at 8.4% interest rate per year compounded monthly, then how long will it take to double?

5.2.2.3 Effective Interest Rate

Activity 5.9

In a group of 5 to 7 students visit any three commercial banks and collect information about annual interest rate, compounding frequencies, fixed time deposit (certificate of deposit) and present the report of your finding to the class.

Whether managing a business or our own personal finances, we are often presented with some choice in options for loans, deposits, and other investments. As borrowers, you want to pay as little interest as possible, and so want to be able to find a low interest rate; as lenders, the shoe is on the other foot and you want to seek out a high interest rate. Assuming all other things like, customer handling, branch locations remain the same, you want to find the best rate you can.

In view of the various ways one can compute interest, it is useful to have a method to compare investment strategies. One such device is the **effective interest rate** r_e defined as the simple interest rate that produces the same yield in one year as compound interest.

For any given interest rate and compounding frequency, there is always an equivalent annually compounded rate.

Definition 5.3

The annually compounded rate which produces the same results as a given interest rate and compounding is called the equivalent annual rate (EAR) or the effective interest rate. The original interest rate is called the nominal rate.

Let Birr 100 is compounded daily for one year with interest rate 8.98% the future value is Birr 109.394. The equivalent effective interest rate is calculated as:

$A = 100(1 + r_e)^1 = 109.394,$ an exponent of 1 has nothing to do.

$100(1 + r_e) = 109.394$. Dividing both sides by 100 we obtain

$$1 + r_e = 1.09394,$$

And subtracting 1 from both sides gives us, $r_e = 0.09394 = 9.39\%$ rounded to two decimal places. This number tells us that 9.39% interest rate compounded annually has the same future value as 8.98 % compounded daily.

Example 1

Find the equivalent annual rate for 9.5% compounded monthly.

Solution

Following the procedure given above:

$A = 100(1 + \frac{0.095}{12})^{12} = 109.92$. Suppose r_e is the rate compounded annually and gives a future value of Birr 109.92.

$$109.92 = 100(1 + R)^1 = 100(1 + r_e)$$

Dividing by 100 both sides, you get $1 + r_e = 1.0992$. Subtracting 1 from both sides we have $r_e = 0.0992 = 9.92\%$.

From the above examples, we summarize how to find effective interest rate. Thus, to find the effective rate for a given nominal rate and compounding frequency, simply find the future value of Birr 100 in 1 year using the nominal rate and compounding. The effective interest rate (rounded to two decimal places) will be the same number as the amount of interest earned.

Alternatively, if interest is compounded *m* times a year, then the effective rate must satisfy the equation,

$$P(1+\frac{r}{m})^m = P(1+r_e) \text{ and so } r_e = (1+\frac{r}{m})^m - 1 \qquad 5.5$$

Example 2

Which of these two banks is offering the best CD rate?

Bank	Rate	Compounding
Z	11.5%	Daily
D	11.5%	Monthly

Solution

Both rates are the same, but Z compounds interest daily. Since more frequent compounding means more interest overall, we know that Z will end up paying more interest, and so their rate is the better one.

Example 3

Which of these two banks is offering the best CD rate?

Bank	Rate	Compounding
B	8.98%	Daily
Credit and Saving Associations	9.05%	Annually

Solution

On this example, on the one hand, Workers Credit and Saving Associations rate is higher. Yet on the other, B Bank is offering daily compounding. It is possible that B Bank daily compounding will more than make up for its lower rate, and that despite initial appearances B in the end will be offering more interest. Yet it is also possible that B daily compounding is not enough to catch up with Credit and Saving Association higher rate.

You can resolve the question by finding the effective rate for each case.

For B bank, by using Equation 5.5 , $r_e = (1+\frac{r}{m})^m - 1$

$$r_e = \left(1 + \frac{0.0898}{365}\right)^{365} - 1 = 1.09394 - 1 = 0.09394.$$

Hence, the effective interest rate of the B bank is 9.394% . Comparing with the credit and saving association rate 9.05%. The rate given by B bank daily compounding wins by a small margin with lower rate.

Alternatively, suppose, $P = Birr\ 100, t = 1\ year$, we treat separately on compound interest formula as follows: in B Bank at the end of the year Birr 100 matured to Birr 109.39 rounded to two decimal place.

$$A = 100(1 + \frac{0.0898}{360})^{360} = 109.394.$$

But for Credit and saving associations, Birr 100 matured to Birr 109.05.

$$A = 100(1 + 0.0905) = 100(1.0905) = 109.05.$$

From this, we can see that B Bank's daily compounding wins by a small margin with lower rate.

Exercise 5.11

1. Find the effective rate equivalent to 12.5%, compounded semi-annually.
2. Hagos plans on opening a new saving account. Which of the following is offering the highest rate?

Bank	Rate	compounding
Bank A	2.44	Annually
Bank B	2.44	Semi-annually
Bank C	2.44	monthly

3. Which of one of the three is offering the best CD rate?

Institute	Rate	Compounding
Bank B	8.98%	monthly
Credit and Saving Associations	9.05%	quarterly
XY Microfinance	9.1%	Semi-annually

4. Dechase has a life insurance policy with N insurance. The company credits interest to his policy's cash value and offers Dechase the choice of two different options:

Option	Rate	Compounding
Daily dividends	8.00%	Daily(bankers rule)
Annual Advancement	8.33%	Annually

Which option would give Dechase the most interest?

5. How much interest would a credit union pay on Birr 475 deposit for 300 days if the effective interest rate was 7%?

6. Find the future value of Birr 372,000 at a 15.25% effective rate for 9 months.

5.2.3 Annuity

Suppose you borrow Birr 1,400,500 at 12% compounded monthly for 5 years to buy an apartment. The deal is unlikely to be that you get Birr 1,400,500 up front, do nothing for 5 years, and then repay the entire loan at the end, all in a single lump sum. Such an arrangement would not make sense either to you or to the lender.

Instead, the usual arrangement requires you pay off the loan by making payments every month. A timeline for your house loan would instead look more like this:

This loan's consistent and regular payments are an example of an annuity. Now define new terminology.

Definition 5.4

An **annuity** is any collection of equal payments made at regular time intervals.

Annuities are very common in business and personal finance. A number of examples are given below; with a few moments' thought you can probably come up with many others.

Some examples of Annuities:

- Monthly payments on a car loan
- Monthly payments on a condominium loan (mortgage).
- Rent payments (until the rent changes)
- Pension on public organization and private company workers (as long as payments do not change)
- Utility bills (same amount paid each month)
- Regularly scheduled deposits to a saving account

Activity 5.10

How is "Iqub" managed in our community? Whether Iqub is annuity or not? Discuss in a group.

In regularly scheduled deposit, the annuity payments are made for the purpose of accumulating a sum of money at a future date. In others, such as a mortgage or car loan, the payments are being made in exchange for a sum of money received at the start of the annuity. In both cases, the sum of money involved is important enough to have a name attached to it.

Example 1

Distinguish between present value and future value.

a. Tesfaye borrowed Birr 160,000 to buy a house. To pay off this mortgage loan, he agreed to make payments of Birr 1,735.52 per month for 30 years. Since his mortgage payments are all equal, and are made at regular intervals, they constitute an annuity. Was the amount he borrowed this annuity's present value or future value?

b. A teacher deposits Birr 250 from each month salary into savings account at work. He will keep this up for the next 40 years, at which time he plans to retire, hopefully having accumulated a large balance in his account. Since equal payments are being made into the account at regular intervals, this is an annuity. Is the value of the account a present value or a future value when the teacher retires?

Solution

a. Birr 160,000 was received at the start of the annuity payments. Therefore, it would be the present value of the annuity.

b. The value of the account when the teacher reaches retirement would be the future value.

Timing of the payment

Our definition of an annuity demands that payments be made at regular intervals (annually, monthly, weekly, daily, etc.), but it is silent about the timing of the payments within each interval. In Example b, the teacher agreed to pay Birr 250 monthly for 40 years. This clearly fits the definition of an annuity. But does the payment begin right away (the payment due at the start of each month) or can he wait 30 days before making the first payment (the payment is due at the end of each month)? There are two types of annuities based on timing of payments.

Definition 5.5

An **ordinary annuity** is an annuity whose payments are made at the end of each time period. An **annuity due** is an annuity whose payments are made at the beginning of each time period.

Example 2

Distinguish between ordinary annuities and annuities due.

a. Kong took out a car loan on May 7. Payments will be made monthly. The first payment is not due until June 7 (the second will be due on July 7, etc.). Is this an ordinary annuity or an annuity due?

b. Wintana rents a house in Birr 35,000 per year for the next 6 years. She paid the first Birr 35,000 right away. Her second payment will come a year from now, at the start of the second year, and so on. Is this an ordinary annuity or an annuity due?

Solution

a. Because his first payment is not made until the end of the first month, the second at the end of the second month, and so on, his car payments are an ordinary annuity.

b. Since the payments come at the start of each year, it is an annuity due.

Exercise 5.12

1. Each of the following problems describes an annuity. Determine whether the amount indicated is the annuity's present value or future value.

   a. Mr. Worku just attended a sales presentation held by the investment company that administers his employer's savings plan. According to the sales representative, if he puts just Birr 1,250 into the plan each month he could be a millionaire when he hits retirement age. Is the Birr 1,000,000 projection a present or future value?

b. City administration won a judgment against a former contractor that will require the contractor to pay the city Birr 52,000.00 per year for the next 20 years. Instead of having to make payments for the next 20 years, the contractor offers to pay the county Birr 835,000.00 in one lump sum today. Would this be a present value or future value?

c. Fifteen years ago Tadesse quit smoking, and instead of spending Birr 70.00 a week on cigarettes he put that money in a special savings account each week. Is the value of his account after 10 years is a present or future value?

2. Each of the following problems describes an annuity. Determine whether it is an ordinary annuity or an annuity due. Justify your answer.

a. Beshir got a share of a company, which will pay him Birr 100,000.00 every year for 20 years, starting right away.

b. At the end of each quarter, a private bank pays each of its shareholders a dividend of 200 Birr for each share owned.

c. Every month I deposited from my salary to our organization credit and saving cooperative union account Birr 400.00.

5.2.3.1 Annuities Computation

At this point, you'll begin to develop the mathematical tools need to work with annuities. At first, you will focus just on finding future values, though later on you address present values.

A. The future value of an ordinary annuity

Suppose that you deposit Birr 1,000 at the end of each year into a saving account that earns 7.5% compounded annually. Assuming you keep the payments up, how much would your account be worth in 10 years? Your annual payments constitute an ordinary annuity, and since we are asking about their accumulated value with interest at the end, we are looking for the future value.

There are three different approaches to develop annuity formula, the Chronological Approach, the Bucket Approach and the Annuity Factor Approach.

Approach 1: The Chronological Approach

A natural way to approach this problem is to build up the account value year by year, crediting interest as it comes due and adding new payments as they are made.

At the end of the first year, Birr 1,000 is deposited.

At the end of the second year, Birr 1000(1+0.075) =1000(1.075)=Birr 1075.00.

Total amount at the end of year 2 is 1075+1000=2075 (1000 is the monthly deposit).

You could continue along this way until the end of the 10 years, the results of which can be summarized in the Table 4 below:

Table 4 Chronological approach

Year	Starting Balance	Interest Earned	Deposit	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	75.00	1000.00	2075.00
3	2075.00	155.63	1000.00	3230.63
4	3230.63	242.30	1000.00	4472.92
5	4472.92	335.47	1000.00	5808.39
6	5808.39	435.63	1000.00	7244.02
7	7244.02	543.30	1000.00	8787.32
8	8787.32	659.05	1000.00	10446.37
9	10446.37	783.48	1000.00	12229.85
10	12229.85	917.24	1000.00	14147.09

At the end of year 3, the interest earned, $I = 2075(1 + 0.075) = 2075(1.075) = 155.63$

Ending balance

$$= 2075 + 155.63 + 1000 = 3230.63$$

This chronological approach may be fine in cases where the total number of payments is small, but clearly we have reason to find an alternative.

This approach nicely reflects what goes on in the account as time goes by. But it is obvious that we wouldn't want to use it to find the future value of 30 years of weekly payments.

Approach 2: The Bucket Approach

Suppose now that you opened up a new account for each year's deposit, instead of making them all to the same account. The first payment of Birr 1,000 is placed in the first bucket. This money is kept on deposit from the end of year 1 until the end of year 10, a total of 9 years. So, at the end of the 10^{th} year, its bucket will contain a total of Birr $1{,}000(1.075)^9 =$ Birr $1{,}917.24$.

You do the same for each of the other eight payments, and then put them all back together again at the end for the total:

Table 5. The Bucket approach

Payment from	Payment Amount	Years of interest	Future value
1	1,000.00	9	Birr 1,917.24
2	1,000.00	8	1,783.48
3	1,000.00	7	1,659.05
4	1,000.00	6	1,543.30
5	1,000.00	5	1,435.63
6	1,000.00	4	1,335.47
7	1,000.00	3	1,242.30
8	1,000.00	2	1,155.63
9	1,000.00	1	1,075.00
10	1,000.00	0	1,000.00
	Grand total		Birr 14,147.09

For the second payment of Birr 1,000, it is kept on deposit from the end of year 2 until the end of year 10, a total of 8 years. So, at the end of the 10^{th} year, its bucket will contain a total of Birr $1{,}000(1.075)^8 =$ Birr $1{,}783.4$. Other payments summarized in Table 5 above.

Notice that we arrived at the same total as with the chronological approach as we would have hoped.

Approach 3: The Annuity Factor Approach

There is a third alternative. Suppose that instead of payments of Birr 1,000 the payments were Birr 3,000 three times as much. It makes sense that the future value would then be three times as much. So to find the future value of the Birr 3,000.00 annuity, we wouldn't need to start from scratch, we could just multiply the Birr 1,000 annuity's future value by 3 to get

$FV = 3(14{,}147.09) = 42{,}441.27$. In general, a larger or smaller payment changes the future value proportionately. Exploiting this, we can define the future value annuity factor.

Definition 5.6

For a given interest rate, payment frequency, and number of payments, the future value annuity factor is the future value that would accumulate if each payment were Birr 1.00. We denote this factor with the symbol $S_{n/i}$ where n is the number of payment and i is the interest rate per payment period.

Using Bucket approach, with payment of Birr 1 we get

Table 6. Bucket Approach with Birr 1 payment

Payment from	Payment Amount	Years of interest	Future value
1	1.00	9	1.91724
2	1.00	8	1.78348
3	1.00	7	1.65905
4	1.00	6	1.54330
5	1.00	5	1.43563
6	1.00	4	1.33547
7	1.00	3	1.2230
8	1.00	2	1.15563
	1.00	1	1.07500
10	1.00	0	1.00000
	Grand total		14.14709

Future value of Ordinary annuity

$$S_{\frac{n}{i}} = (1+0.075)^9 + (1+0.075)^8 + \cdots + (1+0.075)^1 + (1+0.075)^0 = 14.14709$$

Since our payments were actually Birr 1,000 times as much, it is logical that the future value we want is:

$$F.V = 1{,}000(14.14709) = 14{,}147.09.$$

Here, $S_{\frac{n}{i}}$ is the Future Value Annuity factor, the formula for future value of ordinary annuity with periodic payment R,

$$FV = R \times S_{\frac{n}{i}} \qquad 5.6$$

A formula for $\mathbf{S}_{\frac{n}{i}}$, future value annuity factor

Let's look back at the bucket approach's calculation of $\boldsymbol{S}_{\frac{n}{i}}$. The values that we added up to get $\boldsymbol{S}_{\frac{n}{i}}$, were:

$$S_{\frac{n}{i}} = (1+0.075)^9 + (1+0.075)^8 + \cdots + (1+0.075)^1 + (1+0.075)^0$$

Multiplying this equation both sides by $(1+0.075)$ you get

$$(1+0.075)S_{\frac{n}{i}} = (1+0.075)^{10} + (1+0.075)^9 + \ldots + (1+0.075)^2 + (1+0.075),$$

$$(1.075)S_{\frac{n}{i}} = (1.075)^{10} + (1.075)^9 + \cdots + (1.075)^2 + (1.075).$$

Suppose we add 1 to both sides to get:

$$(1.075)S_{\frac{n}{i}} + 1 = (1.075)^{10} + (1.075)^9 + \cdots + (1.075)^2 + (1.075) + 1,$$

the right hand side is rewritten as

$$(1.075)S_{\frac{n}{i}} + 1 = (1.075)^{10} + S_{\frac{n}{i}}$$

Subtracting $1 + S_{\frac{n}{i}}$ from both sides we obtain

$(1.075)S_{\frac{n}{i}} - S_{\frac{n}{i}} = (1.075)^{10} - 1$, which leads to

$$S_{\frac{n}{i}}(1.075 - 1) = (1.075)^{10} - 1$$

$$S_{\frac{n}{i}} = \frac{(1.075)^{10} - 1}{0.075}$$

Evaluating this we get, $S_{\frac{n}{i}} = 14.14709$ rounded to 5 digits where $n = 10, i = 0.075 = 7.5\%$. Which is the same annuity factor we calculated earlier.

In general, the future value of annuity factor is

$$S_{\frac{n}{i}} = \frac{(1+i)^n - 1}{i} \qquad 5.7$$

Where i represents the interest rate per payment period and n represents the number of payments.

Activity 5.11

Derive formula (5.7) by using Geometric series concept from Unit 1.

Example 1

How much will I have as a future value if I deposit Birr 5,000 at the end of each year into an account paying 9.5 % compounded annually for 20 years?

Solution

The payments are equal and at regular intervals, and their timing is at the end of each period, so we have an ordinary annuity. Repeat the above process for any interest rate and number of payments. There was nothing special about $n = 10$ and $i = 0.075$. If we have $n = 20$ and $i = 0.095$, the same process would work. So the future value of the annuity factor is given as

$$S_{\frac{n}{i}} = \frac{(1.095)^{20} - 1}{0.95},$$

the future value will be

$$FV = 5000S_{\frac{20}{0.095}}$$

$$= 5000\left(\frac{(1+0.095)^{20} - 1}{0.095}\right) = 270{,}611.16.$$

Example 2

Suppose that Birr 750 is deposited each year into an account paying 12 % interest compounded annually. What will the future value of the account be for 30years?

Solution

Since the payments are equal and made at regular intervals, this is an annuity; since the timing of the payments is unspecified, we assume it to be an ordinary annuity. By using formula 5.5 and the annuity factor formula 5.6 for the periodic payment R=750,

$$FV = RS_{\frac{n}{i}} \quad = 750\left(\frac{(1+0.12)^{30}-1}{0.12}\right)$$

This value is calculated by using calculator as

Operation	$1 + 0.12 =$	^30	-1	/12	*750
Result	1.12	29.9600	28.9600	241.3327	180,999.51

After 30 years the depositor will have Birr 180,999.51.

Example 3

Find the future value of quarterly payments of Birr 750 for 5 years, assuming a 12% interest rate.

Solution

We first determine the values of i and n. This is done exactly in the same way we did it with compound interest in Section 5.2. Here $r = 0.12,\ m = 4$ since the payment made quarterly

$i = \frac{r}{m} = \frac{0.12}{4} = 0.03$ and $n = m \times t = 4 \times 5 = 20$.

This implies 20 payments of Birr 750 will be made in five years.

Before we proceed, we need to find the annuity factor. To obtain the factor by using the formula, substitute in these values of n and i to get:

$$S_{\frac{n}{i}} = \left(\frac{(1+i)^n - 1}{i}\right) = \left(\frac{(1+0.03)^{20} - 1}{0.03}\right)$$

The annuity factor

$$S_{\frac{20}{0.03}} = 26.87037.$$

Now that we have the annity factor, we can complete the calculation. Plugging into our annuity formula (5.5), we get

$FV = 750 \times S_{\frac{n}{i}} = 750 \times 26.87037 = 20{,}152.778.$

Exercise 5.13

1. Ato Tesfaye works in XYZ-company earning a monthly salary of Birr 10,460. He is also a member of the credit and saving cooperative union of his company and deposits 25% of his monthly salary at the end of each month at 7% compounded monthly.
   a. Compute Ato Tesfaye accumulated balance by the end of three years.
   b. How much interest has he earned?
2. You are 25 years old and decide to start saving for your retirement. You plan to save Birr 15,000 at the end of each year (so the first deposit will be one year from now), and will make the last deposit when you retire at age 65. Suppose you earn 8% per year on your retirement savings.
   a. How much will you have saved for retirement?
   b. How much will you have saved if you wait until age 35 to start saving (again, with your first deposit at the end of the year)?
   c. Develop a saving schedule by using the bucket approach.
3. Find the future value annuity factor for a monthly annuity, assuming the term is 15 years and the interest rate is 11.1% compounded monthly.
4. At the end of each month, Rekike deposits Birr 250 into a saving account that pays 12% interest compounded monthly. Assuming that she keeps this up, and that the interest rate does not change, how much will her deposits have grown to after twenty years? How much total interest did Rekike earn?
5. Define the three approaches of calculating annuity and discuss the pros and cons of each approach.

6. Calculate the annuity factor for each of the following cases.

a. If $n = 20, i = 0.025$

b. If $n = 1, i = 0.075$

c. If $n = 60, i = 0.0025$

d. If $n = 1, i = 0.1$

B. The Future Value of an Annuity Due

With an annuity due, payments are made at the beginning of each period rather than the end. Each payment is made earlier, so it stands to reason that an annuity due would have a larger future value than an ordinary annuity, since the payments have longer to earn interest.

To see more, let's revisit the 10 year's Birr 1,000 per year annuity with 7.5% interest, this time, though, as an annuity due. The first payment would earn interest from the start of the first year until the end of the tenth year, for a total of ten years. The second payment would earn interest for nine years, the third payment for eight years, and soon. The process summarized in Table 7.

Table 7. Future value annuity due factor

Payment from	Payment Amount	Years of interest	Future value
1	1,000.00	10	2061.03156
2	1,000.00	9	1917.23866
3	1,000.00	8	1783.47783
4	1,000.00	7	1659.04914
5	1,000.00	6	1543.30153
6	1,000.00	5	1435.62933
7	1,000.00	4	1335.46914
8	1,000.00	3	1242.29688
9	1,000.00	2	1155.625
10	1,000.00	1	1075
	Grand total		15,208.11906

Notice that this is the same table we used for the ordinary annuity, except that every payment is being credited with 1 additional year of interest. Since crediting 1 year's extra interest is equivalent to multiplying everything by 1.075, we get our future value by multiplying the ordinary annuity's future value by 1.075.

Trying it out, we see that Birr 14,147.09 (1.075) does indeed equal Birr 15,208.11906.

The future value of annuities due is calculated by the formula:

$$FV = R \times S_{\frac{n}{i}}(1+i) \qquad 5.8$$

where R is the periodic payment.

Example 4

On New Year's Day 2012, Mamo resolved to deposit Birr 2,500 at the start of each year into a retirement savings account. Assuming that he sticks to this resolution, and that his account earns 8.75% compounded annually, how much will he have after 40 years?

Solution

The payments are equal and made at the start of each year, so this is an annuity due. Thus,

$$FV = R \times S_{\frac{n}{i}}(1+i) = 2500 \times S_{\frac{n}{i}}(1+0.0875)$$

Before going any further, we must find the annuity factor. Using Formula (5.7) you get

$$S_{\frac{n}{i}} = \left(\frac{(1+i)^n - 1}{i}\right) = \left(\frac{(1+0.0875)^{40} - 1}{0.0875}\right) = 316.0346817$$

Returning to the FV formula by multiplying annuity factor, we get:

$FV = 2500 \times 316.0346817 \times (1+0.0875) = 859{,}219.2911.$

So if Mamo does stick with the program, he will have Birr 859,219.30, or nearly Birr 860,000 after 40 years.

Exercise 5.14

1. Suppose that you deposit Birr 3,500 into a saving account today, and vow to do the same on this date every year. Suppose that your account earns 12.6% compounded monthly and assume there is no drawing from this account. How much will your deposits grow to in thirty years?
2. Abduilkadir has "Iqub". He will pay Birr 1000 at the beginning of each month for the next five years. How much will he get at the end of the term?
3. Suppose that Ato Kifle deposits Birr 1000 into a saving account on behalf of his daughter at the beginning of each month with annual interest rate of 8.4% compounded monthly for the next five years. How much interest will she have after five years? Compare and discuss this result with problem 2.

5.2.3.2 Present Values of Annuity

So far, you have considered annuities whose payments and interest build up toward a future value. Now, you will develop the mathematical tools to deal with annuity present values.

Definition: For a given interest rate, payment frequency, and number of payments, the present value annuity factor is the present value of an annuity at this rate, payment frequency, and number of payments if each payment were Birr 1. You denote this factor with the symbol $a_{\overline{n}|i}$ where n is the number of payments and i is the interest rate per payment period.

You will use these present value annuity factors in much the same way as we did future value annuity factors.

Present value of ordinary annuity

$$PV = R \times a_{\overline{n}|i} \qquad 5.9$$

Present value of annuity due

$$PV = R \times a_{\overline{n}|i}(1+i) \qquad 5.10$$

Formulas for the Present Value of an Annuity

Suppose that we know the payments, term, and interest rate for an annuity, and want to determine its present value. We can find the future value that this annuity could accumulate to by using:

$$FV = RS_{\frac{n}{i}}$$

As we've just seen, we can relate this future value to the present value by means of the compound interest formula

$$FV = PV(1+i)^n \qquad 5.11$$

Equating (5.10) and (5.11) we obtain

$$RS_{\frac{n}{i}} = PV(1+i)^n$$

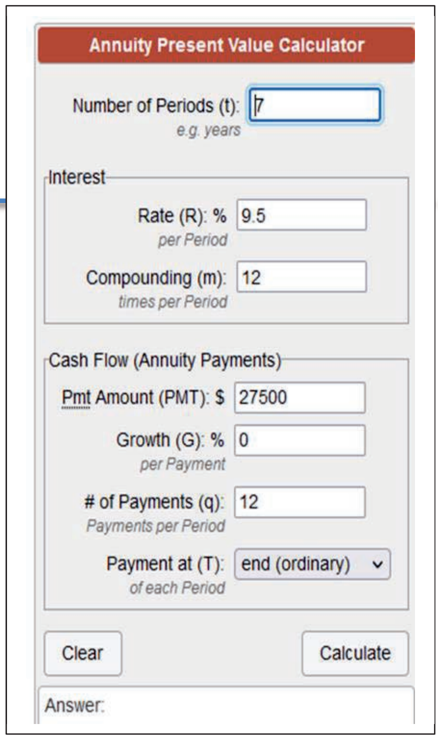
Solving for PV we have

$$PV = R\frac{S_{\frac{n}{i}}}{(1+i)^n} \qquad 5.12$$

Therefore, the present value annuity factor

$$a_{\frac{n}{i}} = \frac{S_{\frac{n}{i}}}{(1+i)^n}, \qquad 5.13$$

Where $S_{\frac{n}{i}}$ is future value annuity factor, n number of periodic payment and i is interest rate of the payment period.



Example 1

Find the present value of Birr 27500 per month for seven years at a rate 9.5% compounded monthly.

Solution

$$i = \frac{0.095}{12} = 0.007917$$

Following our present value formula (5.9) we have:

$PV = R \times a_{\overline{n}|i} = 27500 \times a_{\overline{n}|i}$

To use Formula (5.13) and by the help of a calculators we obtain

$$S_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{(1+0.007917)^{84} - 1}{0.007917} = 118.6636$$

$$a_{\overline{n}|i} = \frac{S_{\overline{n}|i}}{(1+i)^n} = \frac{118.6636}{1.007917} = 61.1838$$

Finally, $PV = 27500 \times 61.1838 = 1{,}682{,}555.93$

The present value $PV =$ Birr 1,682,555.93.

https://www.calculatorsoup.com/calculators/financial/present-value-annuity-calculator.php,

Example 2

Use the annuity factor calculated in Example 1 to find the monthly payment on Birr 800,000 car loan at 9.5% for seven years.

Solution

On this example the required is the monthly payment R.

$$PV = R \times a_{\overline{n}|i}$$

$$800000 = R \times 61.1838$$

Divide both sides by 61.1838

You will get $R = \frac{800000}{61.1838} = 13075.357$. So the monthly payment for the loan Birr 800,000 is around Birr 13,075.

Exercise 5.15

1. Find the present value annuity factor for a five year annuity with quarterly payments and a 9% interest rate.
2. W/ro Aster borrowed Birr 650,000 for 7 years at 9.5% to buy a house. How much is her monthly payment?
3. Find the present value of Birr 13500 for 10 years at a rate of 12% compounded monthly.

5.2.4 Amortization

Activity 5.12

Assume the family head takes a loan of Birr 10,000 for a year. How does he/she will return the loan?

One of the most familiar applications of Time Value of Money (TVOM) concepts is the amortization schedule. Amortization means "The action or process of gradually writing off the initial cost of an asset." An amortization schedule is a list of balances, payments, and interest charges from the inception of a loan until its payoff.

For example, we have a one-year Birr 1,000 loan at 12% compounded monthly. How do we construct a schedule that shows the allocation of payments to principal and interest over the life of the loan?

The first step is to determine the payment amount that will amortize the loan. This can be accomplished either with the TVOM equation or with spreadsheet. We'll start with PV of an annuity equation

$PV = R\frac{S_{\overline{n}|i}}{(1+i)^n} = Ra_{\overline{n}|i}$, replacing $S_{\overline{n}|i}$ by $\frac{(1+i)^n-1}{i}$ we obtain

$$PV = R\frac{\frac{(1+i)^n-1}{i}}{(1+i)^n} = R(\frac{1-(1+i)^{-n}}{i}).$$

In this situation the inputs to the **PV** of an annuity equation are as follows,

$$PV = Birr\ 1000.00,\ \ n = 1 \times 12 = 12, \qquad i = 0.12/12 = 0.01$$

And rearranging the above equations to solve R,

$$R = PV\left(\frac{i}{1-(1+i)^{-n}}\right) = 1000 \times \left(\frac{0.01}{1-1.01^{-12}}\right) = 88.85$$

So the payment amount that will amortize the Birr1, 000 loan is Birr 88.85 (rounded). The table that shows series of payments is known as amortization schedule. For Birr 1000 loan with equal monthly payments for one year at 12% the amortization schedule is shown in Table 8 (rounded in two decimal).

Table 8. Amortization of Birr 1000 with 12% interest rate compounded monthly

Month	Principal	Payment	Interest	Payment to principal
1	1000.00	88.85	10.00	78.85
2	921.15	88.85	9.21	79.64
3	841.51	88.85	8.42	80.43
4	761.08	88.85	7.61	81.24
5	679.84	88.85	6.80	82.05
6	597.79	88.85	5.98	82.87
7	514.91	88.85	5.15	83.70
8	431.21	88.85	4.31	84.54
9	346.67	88.85	3.47	85.38
10	261.29	88.85	2.61	86.24
11	175.05	88.85	1.75	87.10
12	87.96	88.85	0.88	87.97

Example 1

A family has purchased a home with Birr 30,000 down payment and a Birr 300,000 mortgage. The mortgage will be amortized over thirty years with equal monthly payments. The interest rate on the mortgage will be 12% per year. Determine the monthly payment and compile an amortization schedule.

Solution

Based on this data, we would like to determine the monthly mortgage payment and compile an amortization table decomposing each of the monthly payments into interest and payment toward principle.

Down payment means a total payment made at the beginning of a payment to secure a loan.

First, we will express annual data as monthly data. Three hundred and sixty (12×30)months will elapse before the mortgage is fully paid, and the monthly interest rate will be 1%, or 12% divided by 12. Given this monthly data, monthly mortgage payments are determined as follows:

$$R = PV\left(\frac{i}{1-(1+i)^{-n}}\right) = 300000 \times \left(\frac{0.01}{1-1.01^{-360}}\right) = 3085.838$$

So the family will pay monthly Birr 3085.84 for the next 30 years.

In this particular example, because n is large (360), use of spreadsheet will make computations substantially more efficient.

B4 ▾ ⋮ ✕ ✓ fx =B1*(B2/(1-(1+B2)^(-360)))

	A	B	C	D
1	PV	300,000.00		
2	i	0.01		
3	n	360		
4	Monthly Payment	3085.837791		

Some key points about amortization

Beyond being able to fill in the rows and columns of an amortization table, there are some key points worth noting about amortization.

- The amount of each payment is the same, but the split between interest and principal changes with each payment.
- As the balance is paid down, the portion of each payment dedicated to interest declines. A larger share of the early payments will thus go toward interest than later payments.
- As interest's share of the payments decreases, principal's share of the payments increases. So not only does each payment reduce the amount owed, but also the pace of the reduction is accelerating.

Exercise 5.16

1. Solomon and Aster took out a thirty year loan for Birr 600,000 at 15%. Calculate their monthly payments. Complete an amortization table for their first 12 monthly payments.
2. Suppose Solomon and Aster took out a fifteen year mortgage instead, for Birr 600,000 and at a 14% rate. Calculate the monthly payments and produce an amortization table for their first 12 monthly payments.
3. Fill in the missing portions of the amortization table. The loan's initial balance was Birr 25000, payments are monthly, and the interest rate is 9%.

Payment Number	Payment amount	interest amount	Principal	Remaining balance
1	500			
2	500			
3	500			
4	500			
5	500			
6	500			

a. How many months will it take to amortize the loan?

b. How much total interest will be paid during the loan period?

5.2.5 Depreciation

Activity 5.13

You bought a mobile phone with Birr 2500. Suppose you want to sell. What is the price of a mobile phone after using, just 1 day, 1 month, and one year? Is there a price change through time? Why?

The cash value of anything that can be owned will change as time goes by. Some things like real estate and collectibles are expected (or at least hoped) to go up in value over time. We call that increase in something's Birr value price appreciation. Other things, though, become less valuable with use and the passing of time. Computers and electronics become obsolete, used cars command lower prices than new ones and business equipment becomes less valuable as it ages and wears out while improved equipment doing the same work comes on the market. The decline in something's cash value is called depreciation.

Let's see some examples on appreciation and depreciation.

Example 1

According to a gold expert, the market price for a gram of 24 carat gold is Birr 3650. In an interview in a business media, he states that he expects this particular gold to appreciate at 8% per year for the next 10 years. If his prediction turns out to be correct, what will the price be 10 years from now?

Solution

This problem is yet another example of a case where even though the 8% appreciation rate is not compound interest, it is a percentage rate of growth and is mathematically equivalent to compound interest. Therefore, we can use the compound interest formula:

$$FV = PV(1+i)^n = 3650 \times (1+0.08)^{10} = 7880.076$$

The price of a gram of gold is expected to more than double in 10 years from now.

Example 2

Ato Worku just bought a new car for Birr 725,000. According to an online used car pricing service, the value of this car will decline at a 15% annual rate. Assuming this is correct, what will be the car's value over five years?

Solution

This problem in just the same way as the previous example, except that here the rate is understood to be -15%.

$$FV = PV(1 + i)^n = 725000 \times (1 - 0.15)^5 = 321{,}686.35$$

After five years, the price of that car declined to Birr 321,686.35. The following spreadsheet shows how the price of the car is declining each year in the next five years.

Year	Starting Price	Decrease	Ending Price
1	725,000	108,750	616,250.00
2	616,250	92,437.5	523,812.50
3	523,812.5	78,571.88	445,240.63
4	445,240.6	66,786.09	378,454.53
5	378,454.5	**56,768.18**	321,686.35
6	321,686.4	48,252.95	273,433.40
7	273,433.4	41,015.01	232,418.39
8	232,418.4	34,862.76	197,555.63
9	197,555.6	29,633.34	167,922.29
10	167,922.3	25,188.34	142,733.94

As shown in columns 3 and 4, the price is declining at a decelerating rate.

Declining Balance depreciation In year 5, the cost of the material is 378,454.50. The depreciation amount $d = 378{,}454.50 \times 0.15 = 56{,}768.175$.

At the end of first year the car's price depreciated by Birr 108,750, i.e. 15% of the car's original price. In second year the price reduced by Birr 92,437.5 which was 15% of the new price of the car Birr 616,250.00. When we calculate depreciation in this way, assuming that each year's depreciation is a set percent of a decreasing value, the amount of annual depreciation decreases with each passing year. For this reason, this is often referred to as declining balance depreciation.

Exercise 5.17

1. The housing market in Addis Ababa is hot, and a local real estate agent predicted at a meeting of the chamber of commerce that prices would rise by an average of 6.3% per year for the foreseeable future. If the average single-family home costs Birr 1,275,356 today, what would the agent predict in 5 years?
2. In August 2013 the price of an ounce of gold was Birr 640. A commentator on a late-night radio program predicted that the price of gold would rise by an average of 15% per year for the next five years. What price will him projecting in August 2018?
3. Tut bought a boat for Birr 34,750. If the market value declines at a steady 8% annual rate, what will the boat's value be in five years? In ten years?
4. Hayat paid Birr 950,409 for a new truck. She expects that the market value will decline by 11% annually. How much does she expect to be able to sell it for in three years?

Straight line depreciation

As we have seen in the previous discussion, declining balance depreciation is often a very reasonable way to project the actual market price of something. There is another very commonly used method for calculating depreciation, though. With straight-line depreciation we assume that the price declines by the same Birr amount (not the same percent) each year.

In order to determine the amount of this annual price decline, we must determine the period of time over which we expect the depreciation to occur, this is often termed the item's useful life.

Assume that at the end of its useful life the item still has some salvage value, called its residual value or salvage value.

Once you have determined the useful life and salvage value, we then calculate the total amount of depreciation to be taken and divide this by the useful life. This gives the amount the price will decrease each year. We call this amount the rate of (straight-line) depreciation, or the (straight-line) depreciation rate. Which was expressed as

$$\text{Deperciation} = \frac{\text{Initial Value} - \text{Residual Value}}{\text{Useful life in years}}$$

The value of the item after a period of time calculated from this rate is called the item's depreciated value.

Example 3

X company purchased a computer for Birr 20,000. The useful life of the computer is 8 years. The computer is assumed to have no salvage value. Find

a) the straight-line depreciation rate,

b) the depreciated value of the computer after 3 years, and

c) the depreciated value after 9 years.

Solution

a) The computer will lose its full Birr 20,000 initial value in eight years, and since it loses the same amount each year, the depreciation rate is Birr 20,000/8= Birr2500 per year.

b) If the computer's value drops by Birr 2,500 per year for three years, that means it will be worth 20,000-3(2500)=Birr 12,500 at the end of three year.

c) Since the computer's useful life is 8 years, from that time on it has a value of zero. So at nine years, the depreciated value would be Birr 0.

Example 4

Suppose that a retailer buys a cooler for Birr 24,800. The useful life is 12 years, and the residual value is Birr 1,700. If the cooler is put into use 5 months before the end of the year, determine the depreciated values of the cooler at the end of the year and at the end of next year.

Solution

First, we determine the annual depreciation amount:

$$\text{Depreciation amount} = \frac{\text{Initial Value} - \text{Residual Value}}{\text{Useful life in years}} = \frac{24800 - 1700}{12}$$

$$= \text{Birr } 1925 \; per\ year.$$

In the first year, the retailer would take 5/12 of a full year's depreciation, or $\left(\frac{5}{12}\right)(1925) = \text{Birr } 802.08$. So at the end of first year, the depreciated value is $24{,}800 - 802.08 = 23{,}997.92$. In next year, a full year's depreciation would be taken, so the end-of-year depreciated value is

$$23{,}997.92 - 1{,}925 = \text{Birr } 22{,}072.92$$

Before leaving this section we compare the two depreciation methods. Both of the types of depreciation that we have considered are useful in different situations. Having seen how each works, it is worthwhile to compare the two methods both mathematically and in how they are most commonly used.

Let's reconsider the case of Example 2, this time with straight-line depreciation. Suppose we assume a useful life of ten years, and assume that at the end of ten years the car has a residual value of Birr 142,733.94.

$$Depreciation\ rate = \frac{725{,}000 - 142{,}733.94}{10} = 58{,}226.61 \; per\ year$$

Year	Starting Price	Decrease	Ending Price
1	725,000	58,226.61	666,773.39
2	666,773.39	58,226.61	608,546.78
3	608,546.78	58,226.61	550,320.17
4	550,320.17	58,226.61	492,093.56
5	492,093.56	58,226.61	433,866.95
6	433,866.95	58,226.61	375,640.34
7	375,640.34	58,226.61	317,413.73
8	317,413.73	58,226.61	259,187.12
9	259,187.12	58,226.61	200,960.51
10	200,960.51	58,226.61	142,733.90

The slight difference between the 10th year value in the table and the residual value we stated is due to rounding of the straight-line depreciation rate.

Showing this in a table similar to the one in Example 2 with 15% declining balance depreciation, this steady decrease is clearly different from what we saw with percent depreciation.

Table 9. Comparing percent and straight line

Year	Percent	Straight line
Start	725,000.00	725000.00
1	616,250.00	666773.39
2	523,812.50	608546.78
3	445,240.63	550320.17
4	378,454.53	492093.56
5	321,686.35	433866.95
6	273,433.40	375640.34
7	232,418.39	317413.73
8	197,555.63	259187.12
9	167,922.29	200960.51
10	142,733.94	142733.90

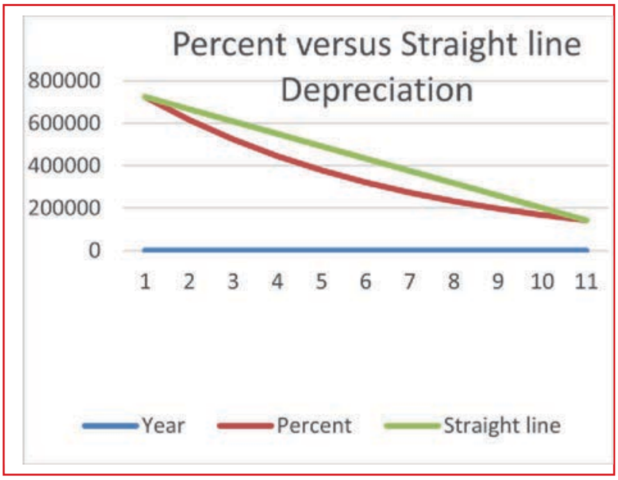


Figure 5.5 Comparing straight line and Declining balance Method

You can also illustrate this comparison with a graph (which also illustrates where the name straight-line depreciation comes from), as seen in the Figure above.

While both methods reflect the same reality, which the value of the car is declining with time, there are some significant differences in the values that they predict.

Exercise 5.18

1. A car that cost Birr 621,450 now is worth Birr 411,050, five years later. What percent depreciation rate would lead to this value?
2. A new refrigerator for a restaurant costs Birr 16,000. The useful life is nine years, and the salvage value is Birr 1,500.
   a. Determine the annual depreciation amount using straight-line depreciation.
   b. Determine the depreciated value of the refrigerator after four years of use.
3. The registrar office bought new filing cabinets for the home office. The total cost was Birr 47,500. The useful life is eight years, and the salvage value is Birr 7,500 . Find the amount of straight-line depreciation per year, and determine the depreciated value of these cabinets after five years of use.

5.3 Saving, Investing and Borrowing Money

"Money is the root of all evil"—so the old adage goes. Whether we agree with that sentiment or not, we have to admit that if money is an evil, it is a necessary one. Love it or hate it, money plays a central role in the world and in our lives, both professional and personal. We all have to earn a living and pay bills, and to accomplish our goals, whatever they may be, reality requires us to manage the financing of those goals.

5.3.1 Saving

Activity 5.14

Discuss why saving is necessary. Which item can we save from the following? Water, natural resource, money or anything what we have.

What is Money?

We may define money in terms of functions it performs. Money is something that people use every day. We earn it and spend it but don't often think much about it. Economists define money as any good that is widely accepted as final payment for goods and services. It is defined as "anything which is generally accepted as a medium of exchange in the settlement of all transactions including debt and acts as a measure and store of value".

Activity 5.15

Why is Birr said to be money?

There are basically two ways to make money.

1. You work for money. Someone pays you to work for them or you have your own business.
2. Your money works for you. You take your money and you save or invest it.

From the above definitions we derive the following functions of Money:

- Money is a medium of exchange. This means that money is widely accepted as a method of payment. When I go to the market, I am confident that the cashier will accept my payment of money. In fact, Ethiopian paper money carries this statement: "Payable to the bearer on demand" or "**ላምጭው እንዲከፈል ሕግ ያስገድዳል**". This means that the Ethiopian government protects my right to pay with Ethiopian Birr.
- Money is a store of value. If I work today and earn Birr 250, I can hold on to the money before I spend it because it will hold its value until tomorrow, next week,

or even next year. In fact, holding money is a more effective way of storing value than holding other items of value such as corn, which might rot. Although it is an efficient store of value, money is not a perfect store of value. Inflation slowly erodes the purchasing power of money over time.

- Money is a unit of account**.** You can think of money as a yardstick - the device we use to measure value in economic transactions.

Figure 5.6 Currently in use Cash note of FDRE

Once we understand meaning and functions of money, we will see reasons for saving of Money.

Reasons for saving of Money

There are a variety of reasons to begin saving money. Different people save for different reasons. Here are seven reasons that you may consider for saving your money.

a. Emergency funds
b. A new car, vacations and other luxury items
c. Maximize Interest Rates
d. Retirement
e. Sinking funds
f. A down payment on a house
g. Your education

Activity 5.16

Form a group and investigate the following issues: Consider the family of each member in your group and let each student ask his/her family.

1. Whether they save money or not.
2. If yes, why do they save?

After collecting these data allocate each answer under the above reasons for saving.

Planning a saving program

Sometimes the hardest thing about saving money is just getting started. This step-by-step guide for how to save money can help you develop a simple and realistic strategy, so you can save for all your short- and long-term savings goals.

1. Record your expense;
2. Budget for saving;
3. Find ways you can reduce your spending;
4. Set saving goals;
5. Decide on your priorities;
6. Pick the right tools;
7. Make saving automatics; and
8. Watch your saving grows.

Activity 5.17

How public and private companies save for:

a. Retirement,

b. down payment for new house.

c. new car and other fixed assets.

Savings as investment

Activity 5.18

Discuss how saved money is converted to investment.

Is there, any other source of capital to start a new business?

While money doesn't grow on trees, it can grow when you save and invest wisely. Small savings add up to big money.

For instance, if you buy a cup of coffee every day for Birr 5, that adds up to Birr 1825 a year. If you saved that Birr 1825 for just one year, and put it into a savings account or investment that earns 7% a year, it would grow to Birr 2559.66 by the end of five years, and by the end of thirty years, to Birr 13,892.36.

Your "savings" are usually put into the safest places, or products, that allow you access to your money at any time. Savings products include savings accounts, checking accounts, and certificates of deposit.

When you "invest," you have a greater chance of losing your money than when you "save." The money you invest in securities, mutual funds, and other similar investments typically is not federally insured. You could lose your "principal"—the amount you've invested. But you also have the opportunity to earn more money.

Activity 5.19

1. Identify kinds of financial institutions.
2. Describe three main factors in choosing a particular institution for saving.

Saving Institutions

Saving institutions are financial institutions that raise loanable funds by selling deposits to the public. They accept deposits from individuals and firms and use these funds to participate in the debt market, making loans or purchasing other debt instruments such as Treasury bills. Savings institutions include commercial banks, savings and loan associations and credit unions.

Commercial Banks

Measured by total assets, commercial banks are the most important type of financial institutions. They perform the critical function of facilitating the flow of funds from surplus units to deficit units. In Ethiopia Commercial Banks are either publicly or privately owned.

Saving and Loan Association

Savings and loans associations (S & Ls) were originally designed as mutual associations, (i.e., owned by depositors) to convert funds from savings accounts into mortgage loans.

Example: Oromia credit and saving, Amhara credit and saving, Dedebit credit and saving, Omo credit and saving.

Credit unions

Credit unions (CUs) are nonprofit organizations composed of members with a common bond, such as an affiliation with a particular labor union, church, university, or even residential area. Their objective is to serve as an intermediary for those members. For example, different sector employee credit unions.

Because CUs do not issue stock, they are technically owned by the depositors. The deposits are called shares, and interest paid on the deposits is called a dividend. Because CUs are nonprofit organizations, their income is not taxed.

Exercise 5.19

1. Discuss the main difference between saving institutions.
2. Which of the following account type earn more money to depositors on specific amount and time? Explain your answer in detail.
   a. Saving account. b. Certificate of deposit. c. Checking account.
3. Which of the following account type costs the saving institutes more money on specific amount and time? Explain your answer in detail.
   a. Saving Account b. Certificate of deposit c. Checking account
4. Explain step by step, list of how to save money.

5.3.2 Investment

Activity 5.20

Mr. Tesfaye purchased Great Ethiopians Renassianse Dam (GERD) bond in September 2022 at a cost of Birr 100,000. Was Mr. Tesfye's purchase of GERD bond an investment or not? Discuss it.

An investment is an asset that will eventually provide value that exceeds the initial cost. The term investment can apply to almost any asset, including intangible assets such as education. In terms of the stock market, investing typically refers to the purchase of stocks or bonds. These securities are designed to provide an investor with future value that will exceed their initial cost.

This could be because of:

- A change in market conditions
- A change in the overall supply (the demand for the product is more of the overall supply)
- Because of a direct improvement being made (such as with buying real estate and renovating to increase the value).

Investments can also be made in other assets. Investing in real estate, for example, could mean buying an inexpensive property, renovating to increase its value, and then selling or leasing for more than the original cost.

Investor is a person or organization that puts money into financial schemes, property, etc. with the expectation of achieving a profit or to get advantage.

Some examples of investors are;

a) Commercial banks provide loan to customers with a specific interest rate and return period and buy and sell foreign currency.

b) Real estate developers, different service providers ….

c) A person who learn, a family who teach and nurse the childeren and so on.:

> "The best investment a person can make is in their own ability". Warren Buffet

5.3.2.1 Investment goal

Activity 5.21

What is your goal in the next five years? Is that an investment goal? Discuss it.

Everybody has investment goals in his/her life, from the old adage of saving for a rainy day to planning a comfortable retirement. Managed funds offer a long-term investment strategy that can help you achieve specific goals. Your choice of fund should reflect the type of life event that you are planning for, as its investment style will determine the returns you can expect over differing timescales.

The three most common types of investment goals are:

- Retirement planning or property purchase over the very long term (15 years or more)
- Life events, such as school fees over the medium term (10-15 years)
- Rainy day or life style funds to finance goals such as to buy car over the medium to shorter term (5-10 years).
- The minimum time horizon for all types of investing should be at least five years.

5.3.2.2 Investment Strategy

The term investment strategy refers to a set of principles designed to help an individual investor achieve his/her financial and investment goals. Your investment strategy depends on your personal circumstances, including your age, capital, risk tolerance, and goals. They can vary from conservative (where they follow a low-risk strategy where the focus is on wealth protection) while others are highly aggressive (seeking rapid growth by focusing on capital appreciation).

Investors can use their strategies to formulate their own portfolios or do so through a financial professional. Strategies aren't static, which means they need to be reviewed periodically as circumstances change.

As mentioned above, people can choose to make their investment decisions on their own or by using a financial professional. More experienced investors are able to make decisions and investment choices on their own. Keep in mind that there is no right way to manage a portfolio, but investors should behave rationally by doing their own research using facts and data to back up decisions by attempting to reduce risk and maintain sufficient liquidity.

Because investment strategies depend so heavily on your personal situation and goals, it's important for you to do your research before you commit your capital to any investment.

Special Considerations.

Risk is a huge component of an investment strategy. Some individuals have a high tolerance for risk while other investors are risk-averse. Here are a few common risk-related rules:

- Investors should only risk what they can afford to lose,
- Riskier investments carry the potential for higher returns,
- Investments that guarantee the preservation of capital also guarantee a minimal return.

Example 1

a) A 25-year-old who starts off his/her career and begins saving for retirement may consider riskier investments because he or she has more time to invest and is more tolerant to risk. He/She can also afford to lose some money in the event the market takes a dive because he/she still has time to earn more money. This means he/she can invest in things like stocks and real estate.

b) A 45-year-old, on the other hand, doesn't have a lot of time to put money away for retirement and would be better off with a conservative plan. He or She may consider investing in things like bonds, government securities, and other safe bets.

c) Meanwhile, someone saving for a vacation or home won't have the same strategy as someone saving for retirement. He or She may be better off putting his or her money away in a savings account or a CD for short-term goals like these.

5.3.2.3 Types of Investment

There are many types of investments available on the market; from stocks and bonds to mutual funds.

A. **Stock** (Equity)**:-** Buying stock is like buying a small fraction of a company; it uses your money to fund the business and you get to enjoy a portion of the profits.

Ownership of stock entitles shareholders to four basic rights, unless a specific right is withheld by agreement with the shareholders:

1) Vote. The right to participate in management by voting on matters that come before the shareholders. A shareholder gets one vote for each share of stock owned.
2) Dividends. The right to receive a proportionate part of any dividend. Each share of stock in a particular class receives an equal dividend.

3) Liquidation. The right to receive a proportionate share of any assets remaining after the corporation pays its liabilities in liquidation. Liquidation means to go out of business, sell the assets, pay all liabilities, and distribute any remaining.

4) Preemption. The right to maintain one's proportionate ownership in the corporation. Suppose you own 5% of a corporation's stock. If the corporation issues 100,000 new shares, it must offer you the opportunity to buy 5% (or 5,000) of the new shares. This right, called the preemptive right, is usually withheld from the stockholders.

Classes of Stock

Corporations issue different types of stock to appeal to a variety of investors. The stock of a corporation may be either common and preferred or Par and no-par.

Common and Preferred. Every corporation issues common stock, the basic form of capital stock. Unless designated otherwise, the word *stock* is understood to mean "common stock." Common stockholders have the four basic rights of stock ownership, unless a right is specifically withheld.

Preferred stock gives its owners certain advantages over common stockholders. Preferred stockholders receive dividends before the common stockholders and they also receive assets before the common stockholders if the corporation liquidates. Owners of preferred stock also have the four basic stockholder rights, unless a right is specifically denied.

Par Value and **No-Par**. Stock may be par-value stock or no-par stock. Par value is an arbitrary amount assigned by a company to a share of its stock

No-par stock does not have par value. But some no-par stock has a **stated value**, which makes it similar to par-value stock.

Example 1

In Ethiopia most of the companies known as Share Company have been established by selling stocks. In 2020 Dashen bank had a share capital of Birr 3,479,224,000 and Awash Bank had Birr 5,848,271,000. This means Dashen Bank has sold 3,479,224 shares and Awash Bank sold 5,848,271 shares, each of them having a par value of Birr 1000.00.

Remark

- Authorized stock is the maximum number of shares the company can issue under its present charter.
- Issued stock is the number of shares the company has issued to its stockholders.
- Stock in the hands of a stockholder is said to be outstanding. The total number of shares of stock outstanding at any time represents 100% ownership of the corporation.
- Stockbroker: A stockbroker is a financial professional who executes orders in the market on behalf of clients. A stockbroker may also be known as a registered representative (RR) or an investment advisor.
- A stock exchange, securities exchange, is an exchange where stockbrokers and traders can buy and sell securities, such as shares of stock, bonds, and other financial instruments. Stock exchanges may also provide facilities for the issue and redemption of such securities and instruments and capital events including the payment of income and dividends.

Example 2

A certain company earned Birr 743,000 in the last quarter, and the company's management has declared a dividend of Birr 450,000. The company has 1,000,000 shares of stock issued. If you own 200 shares of the company's stock, how much will you receive as a dividend?

Solution

The Birr 450,000 total dividend must be distributed among the shareholders based on the number of shares each one owns; 450,000/1,000,000 shares works out to Birr 0.45 per number of share. Since you own 200 shares, you will receive (200 shares)(0.45 per number of share) =Birr 90.00.

Exercise 5.20

1. Explain the main difference between common and preferred shares.
2. List out the basic rights and responsibilities of shareholders.
3. XY Enterprise is a corporation with 3,000 shares. For the current quarter, the company will pay out dividends totaling Birr 37,560.
   a. Find the dividend per share.
   b. If Fantu owns 1,445 shares, how much will she receive?

B. **Bonds/Debt Instruments**: Buying bonds is like having a company takes out a loan from you. It will pay you interest on the principal as well as the full amount later on.

For Example, GERD bond. Here investors are anyone who buy bonds, borrower (issuer of the bond) is Ethiopian Government.

Before proceeding any further, we need to define certain terms that are commonly used with bonds. The issuer of the bond is the entity that creates the bond as a means to borrow funds. Simply, the issuer is the borrower. The par value of a bond is the amount that will be paid to the bond's owner at maturity. This can also be called the face value of the bond. A bond is said to be redeemed when the borrower pays the bond's owner its par value. For this reason, the par value is also sometimes referred to as the redemption value. Normally, redemption occurs at the bond's maturity date, but when a bond is redeemed early, we say it is called; bonds that may be called are said to be callable.

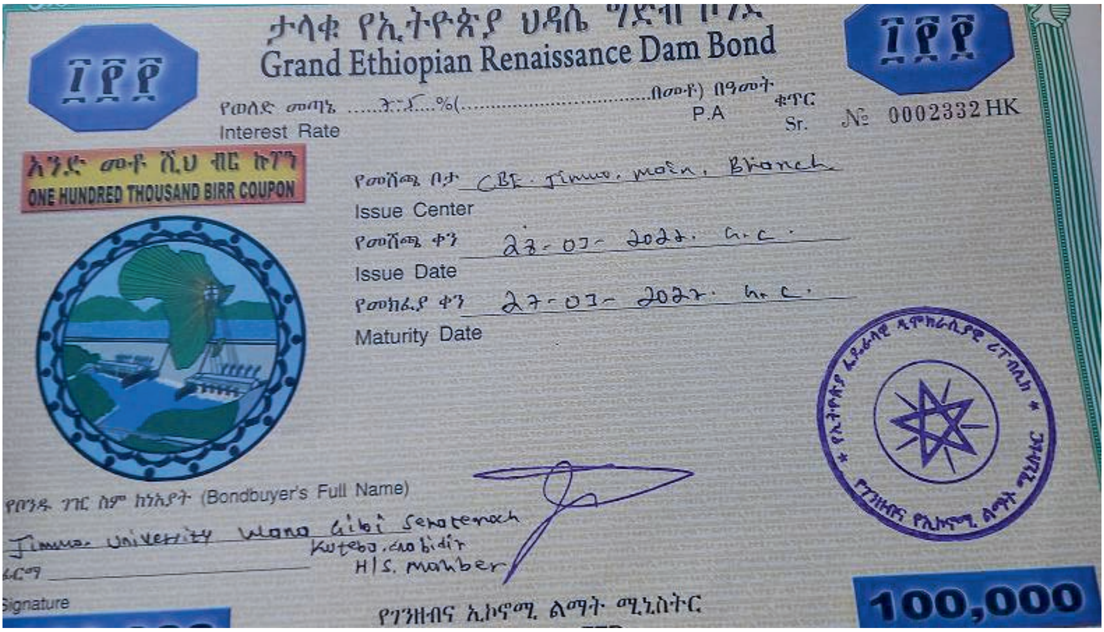


Figure 5.7 GERD Bond

Characteristics of Bonds

Most bond certificate have some common basic characteristics including:

- **Face value** is the money amount the bond will be worth at maturity. It is also the reference amount the bond issuer uses when calculating interest payments. For example, say an investor purchases a bond at a premium Birr 1,090 and another investor buys the same bond later when it is trading at a discount for Birr 980. When the bond matures, both investors will receive the Birr 1,000 face value of the bond.
- **The coupon rate** is the rate of interest the bond issuer will pay on the face value of the bond, expressed as a percentage. For example, a 5% coupon rate on Birr

1000 bond means that bondholders will receive $0.05 \times 1000 =$ Birr 50 every year.

- **Coupon dates** are the dates on which the bond issuer will make interest payments. Payments can be made in any interval.
- **The maturity date** is the date on which the bond will mature and the bond issuer will pay the bondholder the face value of the bond.
- **The issue price** is the price at which the bond issuer originally sells the bonds.

Activity 5.22

Find an image of a GERD bond and check if all the above characteristics are included on the GERD bond.

Example 3

On May 25, 2017, the Ethiopian government issued a Birr 1,000 par value bond with an 8% coupon rate and May 25, 2022, which is maturity date. Interest will be paid semiannually. What payments will the owner of this bond receive?

Solution

The owner of this bond will receive the par value, Birr 1,000, on the maturity date of May 25, 2022. Interest payments will be made semiannually, on November 25 and May 25 of each year until maturity. The amount of the interest payments will be

$$I = PRT = 1000 \times 0.08 \times \frac{1}{2} = \text{Birr } 40.$$

The owner of this bond will collect semiannually Birr 40 until maturity date.

C. **Mutual Funds**

Mutual funds pool money from several investors and invest it in different asset classes. Each investor sees a return based on performance and the size of their initial contribution. Investors buy shares in mutual funds. Each share represents an investor's part ownership in the fund and the income it generates.

5.3.2.4 Return on Investment

Return on investment, or ROI, is a metric that evaluates approximately how much value has been gained from an investment relative to the cost. For example, if you had purchased an asset for Birr 10,000.00 and the value appreciates to Birr 12,000.00, then you have gained Birr 2000.00 worth of value for an ROI of 20%.

Return on investment is calculated by the following formula:

$$ROI = \frac{\text{Current Value of investment-cost of investment}}{\text{cost of investment}}$$

Example 1

You bought an asset currently worth Birr 1000. You initially purchased the asset for Birr 800. The return on investment for that asset would be:

$$ROI = \frac{\text{Current Value of investment-cost of investment}}{\text{cost of investment}} = \frac{1000-800}{800}$$

$$= 25\%$$

Exercise 5.21

1. Explain the main difference between bond and stock.
2. The coupon rate for a Birr 1,000 par value bond is 7.5%. Find the semiannual interest payment for this bond.
3. A government bond with five years to maturity carries a 6% coupon rate for a Birr 1,000 par value. Find monthly interest payment for this bond. The agent bank calculates interest monthly with 5% annual interest rate on deposit. What payment will receive the owner of the bond at maturity date?
4. Initial investment of Z Company is Birr 2,000,000. After five years the investment grows to Birr 3.5 million. Calculate return on investment of Z Company.

5.3.3 Borrowing Money

Activity 5.23

Discuss the following in a group.

a. how one finance his/her business;

b. source of finance for business;

c. institutions that finance the business;

d. the advantages and disadvantages of different sources of finance.

Financing is the process of providing funds for business activities, making purchases, or investing. Financial institutions, such as banks, are in the business of providing capital to businesses, consumers, and investors to help them achieve their goals. The use of financing is vital in any economic system, as it allows companies to purchase products out of their immediate reach.

To raise capital for business needs, companies primarily have two types of financing as an option: equity financing and debt financing.

Equity financing

Equity financing involves selling a portion of a company's equity in return for capital. For example, the owner of company ABC might need to raise capital to fund business expansion. The owner decides to give up 20% of ownership in the company and sell it to an investor in return for capital. That investor now owns 20% of the company and has a voice in all business decisions going forward.

The main advantage of equity financing is that there is no obligation to repay the money acquired through it. Equity financing places no additional financial burden on the company. Since there are no required monthly payments associated with equity financing, the company has more capital available to invest in growing the business.

The drawback of equity financing is:

- The owner has to give investors an ownership percentage of the company
- You have to share your profits with investors
- You give up some control over your company
- It may be more expensive than borrowing.

Example 1

Issuing a share is an example of equity financing.

Debt Financing

Debt financing involves the borrowing of money and paying it back with interest. The most common form of debt financing is a loan. Debt financing sometimes comes with restrictions on the company's activities that may prevent it from taking advantage of opportunities outside the realm of its core business.

The advantages of debt financing are numerous. First, the **lender** has no control over your business. Once you pay the loan back, your relationship with the financier ends. Next, the interest you pay is tax deductible. Finally, it is easy to forecast expenses because loan payments do not fluctuate.

The downside to debt financing is very real to anybody who has debt. Debt is a bet on your future ability to pay back the loan.

Example 2

Consider you are an owner of small business company and need Birr 40,000 to finance the business. You can either take out a Birr 40,000 bank loan at a 12% interest rate, or you can sell a 30% stake in your business to your neighbor for Birr 40,000. Which one is a better decision for you?

Solution

Suppose your business earns a Birr 20,000 profit during the next year. If you took the bank loan, your interest expense (cost of debt financing) would be Birr 4,800, leaving you with Birr 15,200 in profit.

Conversely, had you used equity financing, you would have zero debt (and as a result, no interest expense), but would keep only 70% of your profit (the other 30% being owned by your neighbor). Therefore, your personal profit would only be Birr 14,000, or (70% of Birr 20,000). Therefore, in this scenario taking bank loan is a better decision.

Source of loan

The main sources of loan or debt financing are saving institutions like commercial banks, saving and loan associations and credit unions. Others include consumer finance companies, insurance companies and private companies.

Types of loan

Based on the security, a loan may be classified as secured and unsecured loan.

Secured loan

With a secured loan, the lender has the right to force the sale of the asset against which the loan is secured if you fail to keep up the repayments. The most common form of secured loan is called a ‘further advance’ and is made against your home by borrowing extra on your mortgage. Secured loans are mostly suitable for borrowing large amounts of money over a longer term, for example, for home improvements, expansion of a business.

Unsecured loan

An unsecured loan means the lender relies on your promise to pay it back. They're taking a bigger risk than with a secured loan, so interest rates for unsecured loans tend to be higher. Unsecured loans are often more expensive and less flexible than secured loans, but suitable if you want a short-term loan.

In accordance with the returning period a loan grouped as short term (1-5 years, middle term (6-10) years and long term loans greater than 10 years.

Other types of loan are discussed as:

Credit union loan

Credit unions are mutual financial organizations which are owned and run by their members for their members. Once you've established a record as a reliable saver they will also lend you money but only what they know you can afford to repay.

Money lines

Money lines are community development finance institutions that lend and invest in deprived areas and underserved markets that cannot access mainstream finance. They provide money for personal loans, home improvements, back to work loans, working capital, bridging loans, property and equipment purchase, startup capital and business purchase.

Overdraft

Overdrafts are like a 'safety net' on your current account; they allow you to borrow up to a certain limit when there's no money in your account and can be useful to cover short term cash flow problems. Overdrafts offer more flexible borrowing than taking out a loan because you can repay them when it suits you, but they're not usually suitable for borrowing large amounts over a long period as the interest rate is generally higher than with a personal loan.

Buying on credit

Buying on credit is a form of borrowing. It can include paying for goods or services using credit cards or under some other credit agreement.

Exercise 5.22

1. A share company needs Birr 20,000,000 to expand the existing company. Then the company looks at the following two sources of finance options:
   a. The company sells 25 % stake by issue 10,000 shares with a value of Birr 1,000 and borrowing the remaining amount from commercial banks.

b. The company borrows all from financial institutions. Assume both options are achievable. At the end of the year the company gets a profit of Birr 5 million. Which option is a better decision for the investor?

2. Write the main advantages and disadvantages of debt financing and equity financing.
3. Investigate company in your local area and explain their source of income to start the business. Categorize the source in to debt and equity financing.
4. Explore sources of loans in your area.
5. When is a loan said to be secured? What is the drawback of unsecured loan?

5.4 Taxation

Activity 5.24

Explain the duties of the government of Ethiopia.

List at least three main duties of the government. Do those duties cost the government? From where does the government get money to cover the cost?

As governments have played a growing role in all economies, they have used increasing amounts of resources for their activities, and taxes have constituted increasing percentages of national income. Either directly or indirectly, the various levels of government provide most education and pay a major proportion of medical bills. They provide national defense, police and fire protection and provide or support a substantial amount of housing, recreation facilities and parklands. They set health standards and ensure adequate water supplies, transportation and other public facilities. They seek to attain a distribution of income regarded as equitable, to stabilize the economy from periods of excessive inflation or unemployment, and to ensure an adequate rate of growth.

According to Richard Musgrave, governmental activities are divided into three parts.

1. **Allocation**: the activities involving the provision of various governmental services to society and thus involving the allocation of resources to the production of these services.
2. **Distribution**: the activities involving in the redistribution of income welfare programs, progressive tax structures and so forth.
3. **Stabilization and growth**: the activities designed to increase economic stability by lessening unemployment and inflation and influencing, if thought desirable, the rate of economic growth.

A tax is a compulsory financial charge or some other type of levy imposed on a taxpayer (an individual or legal entity) by a governmental organization in order to fund government spending and various public expenditures. Taxes consist of direct or indirect taxes and may be paid in money or as its labor equivalent. Taxes are important sources of public revenue.

Ethiopia has a federal tax system, with tax powers and revenues divided between the federal government and the regional states. The power to levy and collect different taxes is allocated either 'exclusively to the federal government; exclusively to the regional states; concurrent to both the federal government and the regional states; [or is] undesignated' (FDRE Constitution, 1995 – Proclamation No. 1/1995).

Concurrent powers are assigned to the federal administration, but the resulting revenue is subject to revenue sharing according to the rules set by the House of the Federation (the higher chamber of parliament). On the other hand, revenues assigned to only one level of government can be fully used at that level, without needing to be shared.

Objectives of taxation

Governments impose and collect taxes to raise revenue. Revenue generation, however, is not the only objective of taxation, though it is clearly the prime objective. Taxes as a fiscal policy instrument are used to address several other objectives such as:

- Removal of inequalities in income and wealth;
- Ensuring economic stability;
- Changing people's behaviors;
- Beneficial diversion of resources;
- Promoting economic growth.

Principles of taxation

The compulsory payment by individuals and companies to the state is called taxation. A government imposes taxes to raise revenue to cover the cost of administration, the maintenance of law and order, defense, education, housing, health, pensions, family allowances, etc. In all these, taxes are imposed to provide revenue to cover government expenditure.

Types of taxes

Taxes are sometimes referred to as direct or indirect. The meaning of these terms can vary in different contexts, which can sometimes lead to confusion. In economics,

Direct taxes refer to those taxes that are paid by the person who earns the income. In direct taxes, the impact and incidence fall on the same person.

By contrast, the cost of **indirect taxes** is borne by someone other than the person responsible for paying them. For example, taxes on goods are often included in the price of the items, so even though the seller sends the payments to the government, the buyer is the real payer. Indirect taxes are sometimes described as hidden taxes because the purchaser of goods or services may not be aware that a proportion of the price is going to the government.

Exercise 5.23

1. List out major activities of the government.
2. What are the main objectives of taxations in Ethiopia? And explain each objective.
3. Why is a tax said to be direct or indirect taxes?

5.4.1 Direct Taxes

According to the Ethiopian tax law direct taxes include all income taxes such as employment income tax, business income tax and land use fee, mining income tax and other income taxes. Generally direct taxes are income based taxes.

Income tax

Income tax is a very important direct tax. It is an important and most significant source of revenue of the government. "Income" means every sort of economic benefit including gains in cash or in kind, from whatever source derived and in whatever form paid, credited or received.

"Taxable income" means the amount of income subject to tax after deduction of all expenses and other deductible items allowed under this Proclamation 286/2002 and Regulations 78/200.

Sources of Income

Income taxable under this proclamation shall include, but not limited to:

- Income from employment;
- Income from business activities;
- Income derived by an entertainer, musician, or sports person from his personal activities;
- Income from entrepreneurial activities carried on by a non-resident through a permanent establishment in Ethiopia;
- Income from immovable property and appurtenances thereto, income from livestock and inventory in agriculture and forestry, and income from usufruct and other rights deriving from immovable property situated in Ethiopia;
- Dividends distributed by a resident company;
- Profit shares paid by a resident registered partnership;
- Interest paid by the national, a regional or local Government or a resident of Ethiopia, or paid by a non-resident through a permanent establishment that he/she maintains in Ethiopia;

- License fees including lease payments, and royalties paid by a resident or paid by a nonresident through a permanent establishment that he/she maintains in Ethiopia.

The above sources of income are grouped under the following four Schedules:

Schedules of Income

The proclamation 979/2016 provides for the taxation of income in accordance with four schedules.

- Schedule 'A' Income from employment;
- Schedule 'B' Income from rental buildings;
- Schedule 'C' Income from business;
- Schedule 'D' Income from other sources including royalties' income from technical services rendered outside the country, income from games of chance, dividend income, casual rental of property, interest income and gains from transfer of investment property.

Schedule 'A': Income from employment

Every person deriving income from employment is liable to pay tax on that income at the rate specified in schedule 'A'. The tax payable on income from employment shall be charged, levied and collected at the following rates. The first Birr 600 (six hundred Birr) of employment income is excluded from taxable income.

Table 10. Employment income tax rate in different taxable income bracket

No.	Taxable Monthly income	Rates of Tax in Percentage
1	Less than Birr 600	Exemption
2	[600-1650] on the next Birr 1,050	10%
3	(1650-3200] on the next Birr 1,550	15%
4	(3200-5250]on the next Birr 2,050	20%
5	(5250-7,800] on the next Birr 2,550	25%
6	(7,800-10,900] on the next Birr 3,100	30%
7	Over 10, 900	35%

Methods of employment income tax computations.

Two methods are used to compute employment income tax.

1. Progression method

The amount of tax is calculated for each layer of tax bracket by multiplying the given rate under schedule A For each additional income.

2. Deduction methods

$$\textbf{Income Tax} = \textbf{Taxable Income} \times \textbf{tax rate} - \textbf{Deduction}$$

It can be explained as follows:

No.	Taxable monthly income(Birr)	Rates of tax in percentage	Deduction (Br.)
1	< 600	Exemption	0
2	(600 – 1,650] on the next 1,050	10%	60
3	(1,650 – 3,200] on the next 1,550	15%	142.5
4	(3,200 – 5,250] on the next 2,050	20%	302.5
5	(5,250 – 7,800] on the next 2,550	25%	565
6	(7,800-10,900] on the next ,3100	30%	955
7	over 10,900	35%	1,500

Deduction is computed by this formula

Deduction = upper taxable income pervious tax bracket tax rate of given bracket cumulative threshold.

Deduction formula illustrated as follow:

$\mathbf{60} = 0.1 \times 600,$

$\mathbf{142.5} = 0.15 \times 600 + 0.05 \times 1050,$

$\mathbf{302.5} = 0.2 \times 600 + 0.1 \times 1050 + 0.05 \times 1550$

$\mathbf{565} = 0.25 \times 600 + 0.15 \times 1050 + 0.1 \times 1550 + 0.05 \times 2050$

Example 1

Assume Mr. Zelalem earns a monthly salary of *Birr* 10,460.00. Calculate income tax of Mr. Zelalem.

Solution

Taxable Income = 10,460.00.

By progression method Birr 10,460 divided into six tax brackets. The summary shown in the following table.

No.	Amount	Rates of tax in percentage	Taxable amount
1	600	exempted	0
2	1050	10%	105
3	1550	15%	232.5
4	2050	20%	410
5	2550	25%	637.5
6	2660	30%	798
Total	10460		2183

Total Income Tax

Progression Method

$$2183 = 0 \times 600 + 0.1 \times 1050 + 0.15 \times 1550 + 0.2 \times 2050 + 0.25 \times 2550 + 0.3 \times 2660.$$

Deduction Method

10,640 lay in tax bracket (7800-10900) the deduction in this interval is 955 and the rate is 30% as shown in the table

Income tax= $0.3 \times 10460 - 955 =$ 3138-955=2183, the result is the same as progression method.

Mr. Zelalem net income after tax is Birr 10,460-2,183=Birr 8,277.00.

Example 2

Assume Mr. Zelalem got a promotion with salary increment of Birr 2,680.00 on the previous amount of Birr 10,460.00.

a. Calculate the net income of Mr. Zelalem.

b. By what percentage did the net income increase?

c. By what percentage did the income tax increase?

Solution

a. The new gross salary of Mr. Zelalem is $10460 + 2680 = 13140.00$. Birr 13,140.00 is in the last tax bracket which is 35% taxable rate. By deduction method $income\ tax = 0.35 \times 13140 - 1500 = 3099$. Net income of Mr. Zelalem is 13140-3099=Birr 10,041.00.(Assume the only deduction is tax)

b. Percentage increase on net income= $\frac{10041-8277}{8277} = 0.213$. which is 21.3%

c. Percentage increase of the income tax is

Percentage increase on income tax= $\frac{3099-2183}{2183} = 0.4196$. which is around 41.96% increase on income tax.

Exercise 5.24

Answer the following questions.

1. Mr. Lodamo earns a basic salary of Birr 14,500. What will be net pay for Mr. Lodamo? Assume there is no deduction other than direct tax.
2. List out all sources of taxable income.
3. What are the main differences between progression and deduction methods?
4. An employee salary increases from Birr 10,000.00 to Birr 21,000.00. What is the percentage change in income tax? What is the percentage change on net income?

Schedule 'B': Tax on Income from Rental of Buildings

Any income arising from rental of buildings is taxable under schedule 'B'. Rental income includes all form of income from rent of a building and rent of furniture and equipment if the building is fully furnished.

In the lease contract there are two parties involved in renting a building, the lessor and the lessee. The party who grants rent of the building is the lessor**.** The one who leases the property for use is the lessee.

Taxable Income

a. Gross income includes all payments, either in cash or benefited in kind, received by the lessor.

b. All payments made by the lessee on the behalf of the lessor.

c. The value of any renovation or improvement to the land or the building is also part of taxable income under this schedule if such cost is borne by the lessee in addition to rent payable.

Deduction

Taxable income from schedule B income is determined by subtracting the allowable deductions from the gross income. Allowable deductions include the following:

A. For lessors that do not maintain books of accounts

- taxes paid with respect to the land and buildings being leased; except income taxes; and
- for taxpayers not maintaining books of account, one fifth (1/5) of the gross income received as rent for buildings, furniture and equipment as an allowance for repairs, maintenance and depreciation of such buildings, furniture and equipment.

B. For lessors that maintain books of accounts

- For taxpayers maintaining books of account, the expenses incurred in earning, securing, and maintaining rental income, to the extent that the expenses can be proven by the taxpayer and subject to the limitations specified by the Proclamation 979/2016, deductible expenses include (but are not limited to) the cost of lease (rent) of land, repairs, maintenance, and depreciation of buildings, furniture and equipment in accordance with Article 23 of this Proclamation as well as interest on bank loans, insurance premiums. i.e. building 5%, computer and related asset 25%, furniture and equipment 20% and other asset 10% of depreciation base.

Tax rate

The tax payable on rented houses shall be charged, levied and collected at the following rates:

(a) if the lessors or owners are bodies, they pay thirty percent (30%) of taxable income,

(b) On income of persons according to the Schedule B (hereunder).

Activity 5.25

Explain the main difference between allowable deductions mentioned in A and B.

Schedule B Tax rate and deduction

No.	Taxable income from rental of building (Income per year)		Tax rate in percentage	Deduction in Birr
	From Birr	To		
1	0	7,200	Exempted	0
2	7,201	19,800	10	720
3	19,801	38,400	15	1,710
4	38,401	63,000	20	3,630
5	63,001	93,600	25	6,780
6	93,601	130,800	30	11,460
7	Over 130,800		35	18,000

Source: Federal Income Tax Proclamation (No. 979/2016).

Example 3

Miss Saba has a building that is available for rent in year 2012. The following are the details of the property let out.

- She has let out for twelve months.
- Actual rent for a month is Birr 30,000
- She paid 15% of the actual rent received as land taxes and 3% as other taxes
- She spent birr 10,000 for maintenance of the building

Other Information in 2012

Type	Original Cost	Additional cost	Total
Building	300,000.00	-	300,000.00
Equipment	150,000.00	-	150,000.00
Computer and accessories	10,000/00	6000.00	16,000.00

Compute the taxable income and tax liability

I) He does not maintain any books of accounts in this regard

II) Assumes that Mr. X has maintained books of accounts.

Solution

I) Annual rental income $12 \times 30{,}000.00 = Birr\ 360{,}000.00$

Allowable deduction

$land\text{ taxes } 360{,}000 \times 0.15 = 54{,}000$

$\text{Other taxes } 360{,}000 \times 0.03 = 10{,}800$

$\text{Maintainance } \left(\frac{1}{5} \times 360{,}000\right) = 72{,}000$

$\text{Total Deduction} = 136{,}800$

$\text{Taxable Income} = 360{,}000 - 136{,}800 = 223{,}200$

Then tax liability should be calculated as; Birr 223,200 is in the tax bracket over 130,800 so the rate is 35% and deduction is also Birr 18,000.

Using this numbers

$$\text{Tax liability} = \text{taxable income} * \text{ tax rate} - \text{ deduction}$$

$$= 223{,}200 \times 0.35 - 18{,}000 = Birr\ 60{,}120.00$$

II) For the existence of a book of account

Depreciation Schedule

- For building $300{,}000 * 0.05 = 15{,}000.00$
- For Equipment $15{,}000 * 0.2 = 3{,}000.00$
- For computer $16{,}000 * 0.25 = 4{,}000.00$

Annual rental income $Birr\ 360{,}000.00$ is the same for both cases.

Allowable deduction

$$\text{land taxes } 360{,}000 \times 0.15 = 54{,}000.00$$

$$\text{Other taxes } 360{,}000 \times 0.03 = 10{,}800.00$$

$$\text{Maintainance } = 10{,}000.00$$

Depreciation expense Building	15,000.00
Depreciation expense Equipment	3,000.00
Depreciation expense computer and accessories	4,000.00
Total Deduction	96,800.00

Taxable Income =Birr 360,000-96,800=Birr 263,200.00

Birr 263,200 is in the tax bracket over 130,800 so the rate is 35% and deduction is also Birr 18,000.

Using this numbers

$$\text{Tax liability} = \text{taxable income} * \text{tax rate} - \text{deduction}$$

$$= 263{,}200 \times 0.35 - 18{,}000 = Birr\ 74{,}120.$$

Liability (generally speaking) is something that is owed to somebody else. Liability can also mean a legal or regulatory risk or obligation.

SCHEDULE 'C' –INCOME FROM BUSINESS

Activity 5.26

What type of business do you know? List out individually and discuss with the classmates.

Business means manufacture or purchase and sale of a commodity with a view to make profit.

The taxable business income of a taxpayer for a tax year shall be determined in accordance with the profit and loss, or income statement, of the taxpayer for the year prepared in accordance with the financial reporting standards, subject to other provisions of the Proclamation 979/2016, Regulations issued by the Council of Ministers, and Directives issued by the Minister.

Business Income Tax Rates

1. The rate of business income tax applicable to a body is [30%].
2. The rates of business income tax applicable to an individual are the same as Schedule B tax rate given above.

Business Category

The Ethiopian tax system classifies businesses into three categories – A, B and C – according to whether the business is incorporated or not, and the size of the business as measured by its turnover. Incorporated taxpayers (corporations) are classified as **Category A** (above 1 million annual turnovers) and face the same tax rate (30%) and administrative requirements regardless of their size. For unincorporated taxpayers, these categories determine the information that firms are required to submit when reporting to the revenue authority.

Category B, Businesses are required to submit a profit and loss statement that summarizes the revenues and expenses of the business over the reporting period, but no balance sheet (financial statement) information is required. Their annual turnover is between Birr 500,000 and 1 million.

Category C, businesses are not required to keep books of accounts, as firms pay their taxes based on an assessment made by the Ministry of Revenue, MoR. Their annual turnover is below Birr 500,000.

Example 4

Melat enterprise, unincorporated business has reported earnings before tax of

Birr 80,000 at the tax year ended June 30, 2013. Determine the amount of business income tax.

Solution

Taxable income Birr 80,000 lies in tax bracket [63,001-93,600] with a rate of 25% and deduction Birr 6,780.00.

Tax payable (business income tax)$= 80,000 \times 0.25 - 6,780 =$ Birr 13,220.00.

Schedule 'D' Income: Other Incomes

Other taxable sources of income include royalties, dividends, interest income, winnings from games of chance, and gains on the disposal of investment assets. Hence, any resident who derives income from these sources (as well as any non-resident who derives income from these sources that are attributable to a permanent establishment) is liable for income tax.

Income source	Applicable tax rate and tax base
Royalty	5% on the gross amount of the royalty
Dividend	10% on the gross amount of the dividend
Interest income	5% of the gross amount of interest in the case of a savings deposit with an Ethiopian financial institution;
Game of chance	15%
Gains on disposal of investment asset	15% for disposal of an immovable asset; 30% for disposal of a share or a bond
Casual rental income	15% on the gross amount of rental income

Example 5

Ato Tolossa leased his personal car for two months at Birr 36,000 per month. Such income is referred to as casual rental income by the tax expert. How much is the tax paid? Who is liable to pay the tax?

Solution

Tax on casual rental of property$=0.15 \times 2 \times 36,000 = 10,800$

The receiver of the income is liable to the tax amount of Birr 10,800.

Example 6

Workers credit and saving associations have 8, 312 common shares. Each has Birr 1000.00 value of Coop Bank. The bank paid Birr 310 per share in the year ended June 30, 2012 E.C.

a) How much dividend is the association entitled to?

b) How much is the tax to be paid?

c) How much did the association earn on the year?

Solution

a) Dividend income= Birr $310 \times 8{,}312 =$ Birr $2{,}576{,}720$.

b) According to tax proclamation 979/2016, dividend income is subject to 10% tax rate.

Tax to be paid=$2{,}576{,}720 \times 0.1 =$ Birr $257{,}672.00$.

c) So the association earn Birr $2{,}576{,}720 - 257{,}672.00 =$ Birr $2{,}319{,}048$.

Exercise 5.25

1. MN Share Company declared to pay 25% dividend to its shareholders. Find the dividend earned and tax to be paid by the following shareholders.
   a. Molla with Birr 300,000 worth of shares.
   b. Jemal with Birr 100,000 worth of shares.
   c. W/ro Gifty with Birr 450,000 worth of shares.
2. If Kelbessa won a lottery worth of Birr 150,000, find the amount of tax he is liable to and his net income.
3. Lapiso rented his loader for 100 days to a contractor at a rate of Birr 15,000 per day. Determine the amount he earns after tax.
4. The top three private commercial banks in Ethiopia gross profit in Birr for 2013 E.C financial year listed below:

Bank	Profit before tax in Birr	Share capital in Birr
Awash	4,823,110,000	8,188,948,000
Dashen	2,426,804,000	4,387,996,000
Abyssinia	2,051,544,000	5,182,212,000

a. Which tax law governs the banking sector?

b. Calculate the tax paid.

c. Calculate the amount earned by each bank.

d. Calculate earnings per share in percentage and explain it.

5. List any five businesses operated in your area and classify as Category-A, B, and C.

5.4.2 Indirect Taxes

Indirect taxes are basically taxes that can be passed on to another entity or individual. They are usually imposed on a manufacturer or supplier who then passes on the tax to the consumer. Many people are not aware of that practically everyone pays taxes, especially indirect taxes. This is because taxes are imposed on almost all the products that we consume. Here are some of the types of indirect taxes.

Activity 5.27

Why are many people not aware of paying tax? Discuss in a group.

Types of indirect taxes

The major types of indirect taxes in Ethiopia are value added tax, turn over tax, excise tax, custom duties and stamp duties.

1. Value added tax (VAT)

This is a sales tax based on the increase in value or price of product at each stage in its manufacture and distribution. The cost of the tax is added to the final price and is eventually paid by the consumer.

The standard rate of VAT in Ethiopia is 15%. However, Ethiopia's tax system also applies differential VAT Zero Rate Tax and some items exempted from VAT.

Example 1

I purchased a product for resale. Unit price from supplier is Birr 100. 15% VAT is applicable. The selling price in my shop includes 20% markup. What is the final VAT amount paid to the government?

Solution

I paid the supplier Birr $100 + 0.15 \times 100 =$ Birr 115, the selling price is calculated with 20% markup, i.e. Birr $100 + 0.2 \times 100 = Birr\ 120$, this price is subject to VAT 15%. The sell price is $120 + 0.15 \times 120 =$ Birr 138. VAT liability of the re-seller is Birr 18-15= Birr 3.00.

Example 2

What is the amount of VAT if the cost of the suit is Birr 7500 (15 % VAT inclusive)?

Solution

$Birr\ 7{,}500$, includes both cost of the suit and the VAT. Setting x is the cost of the suit.

$Birr\ 7500 = x + 0.15\,x = (1.15)x$. Dividing both sides by 1.15 we obtain $x = 7500/1.15 = 6521.74$. Hence, the amount of VAT:

$7500 - 6521.74 =$ Birr 978.26.

2. Turnover tax (TOT)

This is an equalization tax imposed on persons not registered for value-added tax to fulfill their obligations and also to enhance fairness in commercial relations and to complete the coverage of the tax system. The standard TOT rate is 2% on goods sold locally and services rendered locally by contractors, grain mills, tractors and combine harvesters, and 10% on other services.

3. Excise tax

Excise tax was introduced into the Ethiopian tax system through the Excise Tax Proclamation No. 307/2002. Under this original law, the tax base for the computation of excise taxes differed for locally produced goods and imported goods. For locally produced goods, the tax base was defined as the cost of production. For imported goods, the tax base was defined as the sum of the Cost of product, plus insurance and freight costs (CFI) value and the applicable customs duty.

However, the computation and validation of the tax base for locally produced goods (i.e. the cost of production) created administrative challenges for the tax authority. The new excise tax proclamation (No. 1186/2020) issued in 2020 therefore changed the tax base for locally produced goods to the ex-factory price of the goods, albeit excluding VAT, the cost of excise stamps and the cost of returnable containers.

Excise duty rates now range from 0% to 500% (of either the ex-factory price or CIF value plus customs duties) with the tax being applied to certain demand-inelastic and luxury items, as well as to goods that are assumed to have negative externalities (e.g. fuel, alcohol, tobacco). The tax rates applied to different goods are based on Proclamation 1186/2020.

4. Customs duty

Customs duties are levied on goods imported into Ethiopia, are administered and collected by the Customs Commission division of the federal MoR (Ministry of Revenue), and must be paid before the imported items can be released to the importers. Standard tax rates vary between 0% and 35%, with goods often used as raw materials or in capital investment projects typically subject to zero or low rates of duty, and goods generally used as final consumption goods typically subject to high rates of duty.

5. Stamp Duty

Stamp duty was introduced in 1998 and is chargeable on various legal, commercial and financial instruments. The legal instrument which regulates stamp duty in Ethiopia is Stamp Duty Proclamation no. 110/1998 and its amendment proclamation no. 612/2008.

Exercise 5.26

1. What is the amount of VAT if the cost of the suit is Birr 85,100 (15 % VAT inclusive)?
2. Goods are purchased at VAT inclusive price of Birr 166,750. The amount of VAT on this transaction are?
3. A company purchased the following items from a stationery shop.

Item	Quantity	Unit price before VAT (in Birr)	Total price
Laptop computers	3	36,000.00	
Desktop computers	5	24,000.00	
Hard disk	2	9,000.00	
Mathematics reference book	10	180.00	

   i) Complete the table.

   ii) Calculate the total VAT paid.

   iii) What is the total price of the items including VAT?

   iv) If the company wants to pay for the stationery after subtracting a 2% withholding tax before VAT;

      a. What is the amount that will be subtracted for withholding?

      b. What is the amount that the company has to pay for the stationary?

4. A shoe dealer purchased shoes from a shoe company Birr 8,000 worth including VAT. If the dealer sold the shoes for Birr 12,000, find the amount of VAT liable to the dealer.

5. An artist sold his new song to a production company for Birr 1,350,000. What is the royalty tax that should be paid by the artist?
6. Why some listed items exempted from VAT?
7. What do we mean when we talk about VAT at a rate of zero percent?
8. What is the difference between zero percent VAT rate and VAT exempted?
9. List out instruments chargeable with stump duty by referring Proclamation no. 110/1998.

Summary

A **ratio** is a comparison of two numbers by division.

The ratio of a to b can be expressed as $a : b$ or $a \div b$.

A **rate**, like a ratio, is a comparison of two quantities, but the quantities may have different units of measure.

A rate that has a denominator of 1 is called a **unit rate**.

A rate that identifies the cost of an item per unit is called the unit price.

The rate of change of a given quantity given by the relation:

$$\text{Rate of Change} = \frac{\text{amount of change}}{\text{original amount}} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}}$$

A **proportion** is an equation that states that two ratios are equal.

In the proportion, $a : b = c : d$ with $b \neq 0$ and $a \neq 0$, the four numbers are referred as the terms of the proportions. The first and the last terms a and d are called the extremes; the second and third terms b and c are called the means.

For three quantities a, b and c such that $\frac{a}{b} = \frac{b}{c}$ which is equivalent to $b^2 = ac$ is called the mean proportional between a and c.

A **compound proportion** is a situation in which one variable quantity depends on two or more other variable quantities.

Two quantities x and y are said to be an inverse proportion if an increase in x causes a proportional decrease in y and vice-versa.

If x and y are in inverse proportion (vary inversely), $xy = k$ where k is a positive number.

If x and y are in direct proportion (vary directly), $y = kx$ where k is a positive number.

A **percentage** is the numerator of a fraction whose denominator is 100.

The difference between a product selling price and its cost is called markup.

$$\text{Markup} = \text{selling price} - \text{purchase price}$$

Interest is what a borrower pays a lender for the temporary use of the lender's money. Or, in other words: Interest is the "rent" that a borrower pays a lender to use the lender's money. The capital originally invested is called the principal. The principal of a loan is the amount borrowed.

The sum of the principal and interest due (or paid) is called the amount (or future value or accumulated value).

A **debtor** is someone who owes someone else money. A **creditor** is someone to whom money is owed.

The amount of time for which a loan is made is called its term.

The future value of a simple interest: $A = P\,(1 + r\,t)$ where A is the future value, P is the principal, r is the simple interest rate per year, and t is the time in years.

Future value or ending amount A is calculated for compound interest by the formula $A = P(1 + r)^n$, where P is the principal, r is the simple interest rate per year and n is the number of years or compounding periods.

Future value of a compound interest $\quad F.V = P(1 + i)^{mt}$

where $F.V$ is amount or future value, P is principal or present value, $i = \frac{r}{m}$, r is annual or nominal rate, t is time in years, and m is the number of conversion periods per year.

Equivalent annual rate(EAR) or the effective interest rate: is annually compounded rate which produces the same results as a given interest rate and compounding. The original interest rate is called the nominal rate.

An annuity is any collection of equal payments made at regular time intervals.

An ordinary annuity is an annuity whose payments are made at the end of each time period.

An annuity due is an annuity whose payment are made at the beginning of each time period.

Amortization means "The action or process of gradually writing off the initial cost of an asset."

The decline in something's cash value is called *depreciation.*

The increase in something's cash value is called *appreciation.*

A tax is a compulsory financial charge or some other type of levy imposed on a taxpayer (an individual or legal entity) by a governmental organization in order to fund government spending and various public expenditures.

Usufruct is the right to enjoy the use and advantages of another's property short of the destruction or waste of its substance.

Review Exercise

1. What is the ratio of 1.8 km to 900 meters?
2. In a family there are three daughters and a son. What is the ratio of the number of;
    a. Females to the number of people in the family?
    b. Males to the number of females in the family?
3. Allocate a profit of Birr 21,300 of a company among three partners in the ratio of their share of the company $1:2:3$.
4. A group of 15 workers can accomplish a job in 28 days. At the same rate by how many workers can the work be accomplished in 8 days less time?
5. What percent of Birr 52 is Birr 3.12?
6. 8.35% of what amount is Birr 18.37?
7. A 6% tax on a pair of shoes amounts to Birr 102. What is the cost of the pair of shoes?
8. If the average daily wage of a laborer increased from Birr 16.00 to Birr 21.64 in the last three years, what is the rate of increase?
9. A radio recorder sold for Birr 210 has a markup of 25% on the selling price. What is the cost?
10. Ato Alula deposited Birr 3,000 in a saving account that pays 6% interest rate per year, compounded quarterly. What is the amount of interest obtained at the end of seven years? (No deposit or withdrawal was made in these seven years).

11. If you receive 6% interest compounded monthly, about how many years will it take for a deposit at time-0 to triple?

12. Ato Alemu makes regular deposits of Birr 230 at the end of each month for three years. What is the future value of his deposit, if interest rate per year is 9% compounded monthly? What is the amount of interest?

13. At the end of each month Ato Mohammed deposits 10% of his salary in a saving institution that pays annual interest rate of 6% for one year and then 15% for the next three years. If the salary of Ato Mohammed is Birr 1800, find the future value of his deposits at the end of the 4 years.

14. Find the future value annuity factor for a monthly annuity, assuming the term is fifteen years and the interest rate is 7.5% compounded monthly.

15. Find the future value of Birr 857.35 per year for 20 years at 10.5% annual interest rate.

16. A piece of machinery costs Birr 50,000 with an estimated residual value of Birr 7,000 and a useful life of 8 years. It was placed in service on July 1 of the current fiscal year. Determine the accumulated depreciation and book value at the end of the following fiscal year using:

    i) the straight line method.

    ii) the percentage method.

17. A share company needs Birr 10,000,000 to expand the existing company. From the following which option gives better economic advantage for the company?

    a. A share issuing 5,000 shares with a value of Birr 1,000 and borrowing the remaining amount from commercial banks.

    b. Issuing 10,000 shares with par value Birr 1,000.

    c. Borrowing Birr 10,000,000 from commercial banks.

TABLE OF RANDOM SAMPLING

11164	36318	75061	37674	26320	75100	10431	20418	19228	91792
21215	91791	76831	58678	87054	31687	93205	43685	19732	08468
10438	44482	66558	37649	08882	90870	12462	41810	01806	02977
36792	26236	33266	66583	60881	97395	20461	36742	02852	50564
73944	04773	12032	51414	82384	38370	00249	80709	72605	67497
49563	12872	14063	93104	78483	72717	68714	18048	25005	04151
64208	48237	41701	73117	33242	42314	83049	21933	92813	04763
51486	72875	38605	29341	80749	80151	33835	52602	79147	08868
99756	26360	64516	17971	48478	09610	04638	17141	09227	10606
71325	55217	13015	72907	00431	45117	33827	92873	02953	85474
65285	97198	12138	53010	94601	15838	16805	61004	43516	17020
17264	57327	38224	29301	31381	38109	34976	65692	98566	29550
95639	99754	31199	92558	68368	04985	51092	37780	40261	14479
61555	76404	86210	11808	12841	45147	97438	60022	12645	62000
78137	98768	04689	87130	79225	08153	84967	64539	79493	74917
62490	99215	84987	28759	19177	14733	24550	28067	68894	38490
24216	63444	21283	07044	92729	37284	13211	37485	10415	36457
16975	95428	33226	55903	31605	43817	22250	03918	46999	98501
59138	39542	71168	57609	91510	77904	74244	50940	31553	62562
29478	59652	50414	31966	87912	87154	12944	49862	96566	48825
96155	95009	27429	72918	08457	78134	48407	26061	58754	05326
29621	66583	62966	12468	20245	14015	04014	35713	03980	03024
12639	75291	71020	17265	41598	64074	64629	63293	53307	48766
14544	37134	54714	02401	63228	26831	19386	15457	17999	18306
83403	88827	09834	11333	68431	31706	26652	04711	34593	22561
67642	05204	30697	44806	96989	68403	85621	45556	35434	09532
64041	99011	14610	40273	09482	62864	01573	82274	81446	32477
17048	94523	97444	59904	16936	39384	97551	09620	63932	03091
93039	89416	52795	10631	09728	68202	20963	02477	55494	39563
82244	34392	96607	17220	51984	10753	76272	50985	97593	34320

96990	55244	70693	25255	40029	23289	48819	07159	60172	81697
09119	74803	97303	88701	51380	73143	98251	78635	27556	20712
57666	41204	47589	78364	38266	94393	70713	53388	79865	92069
46492	61594	26729	58272	81754	14648	77210	12923	53712	87771
08433	19172	08320	20839	13715	10597	17234	39355	74816	03363
10011	75004	86054	41190	10061	19660	03500	68412	57812	57929
92420	65431	16530	05547	10683	88102	30176	84750	10115	69220
35542	55865	07304	47010	43233	57022	52161	82976	47981	46588
86595	26247	18552	29491	33712	32285	64844	69395	41387	87195
72115	34985	58036	99137	47482	06204	24138	24272	16196	04393
07428	58863	96023	88936	51343	70958	96768	74317	27176	29600
35379	27922	28906	55013	26937	48174	04197	36074	65315	12537
10982	22807	10920	26299	23593	64629	57801	10437	43965	15344
90127	33341	77806	12446	15444	49244	47277	11346	15884	28131
63002	12990	23510	68774	48983	20481	59815	67248	17076	78910
40779	86382	48454	65269	91239	45989	45389	54847	77919	41105
43216	12608	18167	84631	94058	82458	15139	76856	86019	47928
96167	64375	74108	93643	09204	98855	59051	56492	11933	64958
70975	62693	35684	72607	23026	37004	32989	24843	01128	74658
			85812	61875	23570	75754			